

A New Parametrization for Tetrad Gravity.

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Abstract

A new version of tetrad gravity in globally hyperbolic, asymptotically flat at spatial infinity spacetimes with Cauchy surfaces diffeomorphic to R^3 is obtained by using a new parametrization of arbitrary cotetrads to define a set of configurational variables to be used in the ADM metric action. Seven of the fourteen first class constraints have the form of the vanishing of canonical momenta. A comparison is made with other models of tetrad gravity and with the ADM canonical formalism for metric gravity.

Keywords: General Relativity, Tetrad Gravity, Constraint Theory

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I. INTRODUCTION

In this paper we develop a new parametrization of arbitrary cotetrads, whose use implies a simplified form of some of the constraints of tetrad gravity. This will open the possibility to restart the study of the canonical reduction of tetrad gravity. Our motivation is the attempt to arrive at a unified description of the four interactions based on Dirac-Bergmann theory of constraints [1], which is the main tool for the Hamiltonian formulation of both gauge theories and general relativity. Therefore, we shall study general relativity from the canonical point of view generalizing to it all the results already obtained in the canonical study of gauge theories in a systematic way, since neither a complete reduction of gravity with an identification of the physical canonical degrees of freedom of the gravitational field nor a detailed study of its Hamiltonian group of gauge transformations, whose infinitesimal generators are the first class constraints, has ever been pushed till the end in an explicit way.

The research program aiming to express the special relativistic strong, weak and electromagnetic interactions in terms of Dirac's observables [1,2] is in an advanced stage of development [3]. This program is based on the Shanmugadhasan canonical transformations [4]: if a system has first class constraints at the Hamiltonian level ¹, then, at least locally, one can find a canonical basis with as many new momenta as first class constraints (Abelianization of first class constraints), with their conjugate canonical variables as Abelianized gauge variables and with the remaining pairs of canonical variables as pairs of canonically conjugate Dirac's observables ². Putting equal to zero the Abelianized gauge variables defines a local gauge of the model. If a system with constraints admits one (or more) *global* Shanmugadhasan canonical transformations, one obtains one (or more) privileged *global gauges* in which the physical Dirac observables are globally defined and globally separated from the gauge degrees of freedom ³. These privileged gauges (when they exist) can be called *generalized Coulomb or radiation gauges*. Second class constraints, when present, are also taken into account by the Shanmugadhasan canonical transformation [4].

The problem of how to covariantize this kind of canonical reduction is solved by using Dirac reformulation (see the book in Ref. [1]) of classical field theory on spacelike hypersurfaces foliating ⁴ Minkowski spacetime M^4 . In this way one gets parametrized Minkowski field theory with a covariant 3+1 splitting of flat spacetime and already in a form suited to

¹So that its dynamics is restricted to a presymplectic submanifold of phase space.

²Canonical basis of physical variables adapted to the chosen Abelianization; they give a trivialization of the BRST construction of observables.

³For systems with a compact configuration space this is in general impossible.

⁴The foliation is defined by an embedding $R \times \Sigma \rightarrow M^4$, $(\tau, \vec{\sigma}) \mapsto z^\mu(\tau, \vec{\sigma})$, with Σ an abstract 3-surface diffeomorphic to R^3 : this is the classical basis of Tomonaga-Schwinger quantum field theory.

the transition to general relativity in its ADM canonical formulation ⁵. The price is that one has to add as new configuration variables the points $z^\mu(\tau, \vec{\sigma})$ of the spacelike hypersurface Σ_τ ⁶ and then define the fields on Σ_τ so that they know the hypersurface Σ_τ of τ -simultaneity ⁷. Then one rewrites the Lagrangian of the given isolated system in the form required by the coupling to an external gravitational field, makes the previous 3+1 splitting of Minkowski spacetime and interpretes all the fields of the system as the new fields on Σ_τ (they are Lorentz scalars, having only surface indices). Instead of considering the 4-metric as describing a gravitational field ⁸, here one replaces the 4-metric with the induced metric $g_{AB}[z] = z_A^{(\mu)} \eta_{(\mu)(\nu)} z_B^{(\nu)}$ on Σ_τ ⁹ and considers the embedding coordinates $z^{(\mu)}(\tau, \vec{\sigma})$ as independent fields ¹⁰. From this Lagrangian, besides a Lorentz-scalar form of the constraints of the given system, we get four extra primary first class constraints $\mathcal{H}_\mu(\tau, \vec{\sigma}) \approx 0$ implying the independence of the description from the choice of the foliation with spacelike hypersurfaces. Therefore the embedding variables $z^{(\mu)}(\tau, \vec{\sigma})$ are the *gauge* variables associated with this kind of general covariance. In special relativity, it is convenient to restrict ourselves to arbitrary spacelike hyperplanes $z^{(\mu)}(\tau, \vec{\sigma}) = x_s^{(\mu)}(\tau) + b_r^{(\mu)}(\tau) \sigma^r$. Since they are described by only 10 variables ¹¹, we remain only with 10 first class constraints determining the 10 variables conjugate to the hyperplane ¹² in terms of the variables of the system.

If we now consider only the set of configurations of the isolated system with timelike ¹³ 4-momenta, we can restrict the description to the so-called *Wigner hyperplanes* orthogonal

⁵See also Ref. [5], where a theoretical study of this problem is done in curved spacetimes.

⁶The only ones carrying Lorentz indices; the scalar parameter τ labels the leaves of the foliation and $\vec{\sigma}$ are curvilinear coordinates on Σ_τ .

⁷For a Klein-Gordon field $\phi(x)$, this new field is $\tilde{\phi}(\tau, \vec{\sigma}) = \phi(z(\tau, \vec{\sigma}))$: it contains the non-local information about the embedding.

⁸Therefore as an independent field as it is done in metric gravity, where one adds the Hilbert action to the action for the matter fields.

⁹A functional of $z^{(\mu)}$; here we use the notation $\sigma^A = (\tau, \sigma^r)$; (μ) is a flat Minkowski index; $z_A^{(\mu)} = \partial z^{(\mu)} / \partial \sigma^A$ are flat cotetrad fields on Minkowski spacetime with the $z_r^{(\mu)}$'s tangent to Σ_τ .

¹⁰This is not possible in metric gravity, because in curved spacetimes $z_A^\mu \neq \partial z^\mu / \partial \sigma^A$ are not tetrad fields since the holonomic coordinates $z^\mu(\tau, \vec{\sigma})$ do not exist.

¹¹An origin $x_s^{(\mu)}(\tau)$ and, on it, three orthogonal spacelike unit vectors $b_r^\mu(\tau)$ generating the fixed constant timelike unit normal to the hyperplane.

¹²They are a 4-momentum $p_s^{(\mu)}$ and the six independent degrees of freedom hidden in a spin tensor $S_s^{(\mu)(\nu)}$.

¹³ $\epsilon p_s^2 > 0$; $\epsilon = \pm 1$ according to the chosen convention for the Lorentz signature of the metric $\eta^{(\mu)(\nu)} = \epsilon(+ - - -)$.

to $p_s^{(\mu)}$ itself. To get this result, we must boost at rest all the variables with Lorentz indices by using the standard Wigner boost $L^{(\mu)}_{(\nu)}(p_s, \overset{\circ}{p}_s)$ for timelike Poincaré orbits, and then add the gauge-fixings $b_{\vec{r}}^{(\mu)}(\tau) - L^{(\mu)}_{\vec{r}}(p_s, \overset{\circ}{p}_s) \approx 0$. Since these gauge-fixings depend on $p_s^{(\mu)}$, the final canonical variables, apart $p_s^{(\mu)}$ itself, are of 3 types: i) there is a non-covariant *external* center-of-mass variable $\tilde{x}^{(\mu)}(\tau)$ ¹⁴; ii) all the 3-vector variables become Wigner spin 1 3-vectors¹⁵; iii) all the other variables are Lorentz scalars. Only four 1st class constraints are left: one of them identifies the invariant mass of the isolated system, to be used as Hamiltonian, while the other three are the rest-frame conditions implying the vanishing of the *internal* (i.e. inside the Wigner hyperplane) total 3-momentum.

We obtain in this way a new kind of instant form of the dynamics (see Ref. [6]), the *Wigner-covariant 1-time rest-frame instant form* [7,3] with a universal breaking of Lorentz covariance independent from the isolated system under investigation. It is the special relativistic generalization of the non-relativistic separation of the center of mass from the relative motions $[H = \frac{\vec{P}^2}{2M} + H_{rel}]$.

As shown in Refs. [7,8], the rest-frame instant form of dynamics automatically gives a physical ultraviolet cutoff in the spirit of Dirac and Yukawa: it is the *Møller radius* [9] $\rho = \sqrt{-\epsilon W^2}/\epsilon P^2 = |\vec{S}|/\sqrt{\epsilon P^2}$ ¹⁶, namely the classical intrinsic radius of the worldtube, around the covariant non-canonical Fokker-Pryce center of inertia $Y^{(\mu)}$, inside which the non-covariance of the canonical center of mass $\tilde{x}^{(\mu)}$ is concentrated. At the quantum level ρ becomes the Compton wavelength of the isolated system multiplied its spin eigenvalue $\sqrt{s(s+1)}$, $\rho \mapsto \hat{\rho} = \sqrt{s(s+1)}\hbar/M = \sqrt{s(s+1)}\lambda_M$ with $M = \sqrt{\epsilon P^2}$ the invariant mass and $\lambda_M = \hbar/M$ its Compton wavelength. Therefore, the *criticism* to classical relativistic physics, based on quantum *pair production*, concerns the testing of distances where, due to the Lorentz signature of spacetime, one has intrinsic classical covariance problems: it is impossible to localize the canonical center of mass $\tilde{x}^{(\mu)}$ of the system in a frame independent way. Let us remember [7] that ρ is also a remnant in flat Minkowski spacetime of the *energy conditions* of general relativity: since the Møller non-canonical, non-covariant center of energy has its non-covariance localized inside the same worldtube with radius ρ (it was discovered in this way) [9], it turns out that for an extended relativistic system with the material radius smaller than its intrinsic radius ρ one has: i) its peripheral rotation velocity can exceed the velocity of light; ii) its classical energy density cannot be positive definite everywhere in every frame.

Now, the real relevant point is that this ultraviolet cutoff determined by ρ exists also in Einstein's general relativity (which is not power counting renormalizable) in the case of asymptotically flat spacetimes, taking into account the Poincaré Casimirs of its asymptotic

¹⁴It is only covariant under the little group of timelike Poincaré orbits like the Newton-Wigner position operator.

¹⁵Boosts in M^4 induce Wigner rotations on them.

¹⁶ $W^2 = -\epsilon P^2 \vec{S}^2$ is the Pauli-Lubanski Casimir when $\epsilon P^2 > 0$.

ADM Poincaré charges ¹⁷ at spatial infinity. See Ref. [10] for the definition of the rest-frame instant form of ADM metric gravity.

Moreover, the extended Heisenberg relations of string theory [11], i.e. $\Delta x = \frac{\hbar}{\Delta p} + \frac{\Delta p}{T_{cs}} = \frac{\hbar}{\Delta p} + \frac{\hbar \Delta p}{L_{cs}^2}$ implying the lower bound $\Delta x > L_{cs} = \sqrt{\hbar/T_{cs}}$ due to the $y + 1/y$ structure, have a counterpart in the quantization of the Møller radius [7]: if we ask that, also at the quantum level, one cannot test the inside of the worldtube, we must ask $\Delta x > \hat{\rho}$ and this is the lower bound implied by the modified uncertainty relation $\Delta x = \frac{\hbar}{\Delta p} + \frac{\hbar \Delta p}{\hat{\rho}^2}$. This could imply that the center-of-mass canonical non-covariant 3-coordinate $\vec{z} = \sqrt{\epsilon P^2}(\vec{x} - \frac{\vec{P}}{P^{(0)}}\tilde{x}^{(0)})$ [7] cannot become a self-adjoint operator. See Hegerfeldt's theorems (quoted in Refs. [8,7]) and his interpretation pointing at the impossibility of a good localization of relativistic particles ¹⁸. Since the eigenfunctions of the canonical center-of-mass operator are playing the role of the wave function of the universe, one could also say that the center-of-mass variable has not to be quantized, because it lies on the classical macroscopic side of Copenhagen's interpretation and, moreover, because, in the spirit of Mach's principle that only relative motions can be observed, no one can observe it (it is only used to define a decoupled *point particle clock*). On the other hand, if one rejects the canonical non-covariant center of mass in favor of the covariant non-canonical Fokker-Pryce center of inertia $Y^{(\mu)}$, $\{Y^{(\mu)}, Y^{(\nu)}\} \neq 0$, one could invoke the philosophy of quantum groups to quantize $Y^{(\mu)}$ to get some kind of quantum plane for the center-of-mass description. Let us remark that the quantization of the square root Hamiltonian done in Ref. [12] is consistent with this problematic.

In conclusion, the best set of canonical coordinates adapted to the constraints and to the geometry of Poincaré orbits and naturally predisposed to the coupling to canonical tetrad gravity is emerging for the electromagnetic, weak and strong interactions with matter described either by fermion fields or by relativistic particles with a definite sign of the energy. Therefore, we can begin to think how to quantize the standard model in the Wigner-covariant Coulomb gauge in the rest-frame instant form with the Möller radius as a ultraviolet cutoff.

Since our aim is to arrive at a unified description of the four interactions, in this paper we put the basis for the canonical reduction to Dirac's observables of tetrad gravity (more natural than metric gravity for the coupling to fermion fields) and for exploring the connection of Dirac's observables with Bergmann's definition of observables and the problem of time in general relativity [13–15].

Our approach to tetrad gravity (see Refs. [16–27] for the existing versions of the theory) utilizes the ADM action of metric gravity with the 4-metric expressed in terms of arbitrary cotetrads, which are parametrized in a particular way in terms of the parameters of special

¹⁷When supertranslations are eliminated with suitable boundary conditions; let us remark that Einstein and Wheeler use closed universes because they don't want to introduce boundary conditions, but in this way they loose Poincaré charges and the possibility to make contact with particle physics and to define spin.

¹⁸Experimentally one determines only a worldtube in spacetime emerging from the interaction region.

Wigner boosts¹⁹ and cotetrads adapted to Σ_τ ²⁰. The introduction of this new parametrization of arbitrary cotetrads in the ADM Lagrangian allows to get a new Lagrangian for tetrad gravity. Then we study the associated Hamiltonian formulation, we identify its fourteen first class constraints and we evaluate their Poisson brackets.

We shall restrict ourselves to the simplest class of spacetimes to have some chance to have a well posed formulation of tetrad gravity, which hopefully will allow to arrive at the end of the canonical reduction. Refs. [28–30] are used for the background in differential geometry. A spacetime is a time-oriented pseudo-Riemannian (or Lorentzian) 4-manifold $(M^4, {}^4g)$ with signature $\epsilon(+ - - -)$ ($\epsilon = \pm 1$) and with a choice of time orientation²¹. In Appendix A we give a review of notions on 4-dimensional pseudo-Riemannian manifolds, tetrads on them and triads on 3-manifolds, which unifies many results, scattered in the literature, needed not only for a well posed formulation of tetrad gravity but also for the further study of its canonical reduction. Also a review of the action principles used in metric and tetrad gravity is given in Appendix A for completeness.

Our spacetimes are assumed to be:

i) *Globally hyperbolic* 4-manifolds, i.e. topologically they are $M^4 \approx R \times \Sigma$, so to have a well posed Cauchy problem (with Σ the abstract model of Cauchy surface) at least till when no singularity develops in M^4 (see the singularity theorems). Therefore, these spacetimes admit regular foliations with orientable, complete, non-intersecting spacelike 3-manifolds: the leaves of the foliation are the embeddings $i_\tau : \Sigma \rightarrow \Sigma_\tau \subset M^4$, $\vec{\sigma} \mapsto z^\mu(\tau, \vec{\sigma})$, where $\vec{\sigma} = \{\sigma^r\}$, $r=1,2,3$, are local coordinates in a chart of the C^∞ -atlas of the abstract 3-manifold Σ and $\tau : M^4 \rightarrow R$, $z^\mu \mapsto \tau(z^\mu)$, is a global timelike future-oriented function labelling the leaves (surfaces of simultaneity). In this way, one obtains 3+1 splittings of M^4 and the possibility of a Hamiltonian formulation.

ii) *Asymptotically flat at spatial infinity*, so to have the possibility to define asymptotic Poincaré charges [31–36,10]: they allow the definition of a Møller radius in general relativity and are a bridge towards a future soldering with the theory of elementary particles in Minkowski spacetime defined as irreducible representation of its kinematical, globally implemented Poincaré group according to Wigner. We will not compactify space infinity at a point like in the spi approach of Ref. [36].

iii) Since we want to be able to introduce Dirac fermion fields, our spacetimes M^4 must admit a *spinor (or spin) structure* [37]. Since we consider non-compact space- and time-orientable spacetimes, spinors can be defined if and only if they are *parallelizable* [38]. This means that we have trivial principal frame bundle $L(M^4) = M^4 \times GL(4, R)$ with $GL(4, R)$ as structure group and trivial orthonormal frame bundle $F(M^4) = M^4 \times SO(3, 1)$; the fibers of $F(M^4)$ are the disjoint union of four components and $F_o(M^4) = M^4 \times L_+^\uparrow$ (with projection $\pi : F_o(M^4) \rightarrow M^4$) corresponds to the proper subgroup $L_+^\uparrow \subset SO(3, 1)$ of the Lorentz

¹⁹This is suggested by the rest-frame Wigner-covariant instant form approach.

²⁰Which, in turn, depend on cotriads on Σ_τ and on lapse and shift functions.

²¹I.e. there exists a continuous, nowhere vanishing timelike vector field which is used to separate the non-spacelike vectors at each point of M^4 in either future- or past-directed vectors.

group. Therefore, global frames (tetrads) and coframes (cotetrads) exist. A spin structure for $F_o(M^4)$ is, in this case, the trivial spin principal $SL(2, \mathbb{C})$ -bundle $S(M^4) = M^4 \times SL(2, \mathbb{C})$ (with projection $\pi_s : S(M^4) \rightarrow M^4$) and a map $\lambda : S(M^4) \rightarrow F_o(M^4)$ such that $\pi(\lambda(p)) = \pi_s(p) \in M^4$ for all $p \in S(M^4)$ and $\lambda(pA) = \lambda(p)\Lambda(A)$ for all $p \in S(M^4)$, $A \in SL(2, \mathbb{C})$, with $\Lambda : SL(2, \mathbb{C}) \rightarrow L_+^\uparrow$ the universal covering homomorphism. Then, Dirac fields are defined as cross sections of a bundle associated with $S(M^4)$ [30]. Since $M^4 \approx R \times \Sigma$ is time- and space-oriented, the hypersurfaces Σ_τ of simultaneity are necessarily space-oriented and are parallelizable (as every 3-manifold [38]): therefore, global triads and cotriads exist. $F(\Sigma_\tau) = \Sigma_\tau \times SO(3)$ is the trivial orthonormal frame $SO(3)$ -bundle and, since one has $\pi_1(SO(3)) = \pi_1(L_+^\uparrow) = \mathbb{Z}_2$ for the first homotopy group, one can define $SU(2)$ spinors on Σ_τ [39,40].

iv) The non-compact parallelizable simultaneity 3-manifolds (the Cauchy surfaces) Σ_τ are assumed to be *topologically trivial*, *geodesically complete*²² and, finally, *diffeomorphic to R^3* . These 3-manifolds have the same manifold structure as Euclidean spaces [29]: a) the geodesic exponential map $Exp_p : T_p \Sigma_\tau \rightarrow \Sigma_\tau$ is a diffeomorphism (Hadamard theorem); b) the sectional curvature is less or equal zero everywhere; c) they have no *conjugate locus*²³ and no *cut locus*²⁴. In these manifolds two points determine a line, so that the *static* tidal forces in Σ_τ due to the 3-curvature tensor are repulsive; instead in M^4 the tidal forces due to the 4-curvature tensor are attractive, since they describe gravitation, which is always attractive, and this implies that the sectional 4-curvature of timelike tangent planes must be negative (this is the source of the singularity theorems) [29]. In 3-manifolds not of this class one has to give a physical (topological) interpretation of *static* quantities like the two quoted loci. In particular, these 3-manifolds have *global charts* inherited by R^3 through the diffeomorphism. Given a Cauchy surface Σ_{τ_o} of this type and a set of Cauchy data for the gravitational field (and for matter, if present), the Hamiltonian evolution we are going to describe will be valid from τ_o till $\tau_o + \Delta\tau$, where the interval $\Delta\tau$ is determined by the appearance of either conjugate points on $\Sigma_{\tau_o + \Delta\tau}$ or 4-dimensional singularities in M^4 on its slice $\Sigma_{\tau_o + \Delta\tau}$.

v) Like in Yang-Mills case [8], the 3-spin-connection on the orthogonal frame $SO(3)$ -bundle (and therefore triads and cotriads) will have to be restricted to *suited weighted Sobolev spaces* to avoid Gribov ambiguities. In turn, this implies the *absence of isometries* of the non-compact Riemannian 3-manifold $(\Sigma_\tau, {}^3g)$ (see for instance the review paper in Ref. [41]). All the problems of the boundary conditions on lapse and shift functions and on cotriads will be studied in connection with the Poincaré charges in a future paper see however Ref. [10] for the case of metric gravity).

Diffeomorphisms on Σ_τ ($Diff \Sigma_\tau$) will be interpreted in the passive way, following Ref.

²²So that the Hopf-Rinow theorem [29] assures metric completeness of the Riemannian 3-manifold $(\Sigma_\tau, {}^3g)$.

²³I.e. there are no pairs of conjugate Jacobi points (intersection points of distinct geodesics through them) on any geodesic.

²⁴I.e. no closed geodesics through any point.

[13], in accord with the Hamiltonian point of view that infinitesimal diffeomorphisms are generated by taking the Poisson bracket with the first class supermomentum constraints²⁵. The Lagrangian approach based on the Hilbert action, connects general covariance with the invariance of the action under spacetime diffeomorphisms ($Diff M^4$) extended to 4-tensors. Therefore, the moduli space (or superspace or space of 4-geometries) is the space $Riem M^4 / Diff M^4$ [42], where $Riem M^4$ is the space of Lorentzian 4-metrics; as shown in Refs. [43,44], superspace, in general, is not a manifold²⁶ due to the existence (in Sobolev spaces) of 4-metrics and 4-geometries with isometries. See Ref. [46] for the study of great diffeomorphisms, which are connected with the existence of disjoint components of the diffeomorphism group²⁷. Instead, in the ADM Hamiltonian formulation of metric gravity [31] space diffeomorphisms are replaced by $Diff \Sigma_\tau$ ²⁸, while time diffeomorphisms are distorted to the transformations generated by the superhamiltonian 1st class constraint [47,14,48] and by the momenta conjugate to the lapse and shift functions. In the Lichnerowicz-York conformal approach to canonical reduction [49,50] (see Refs. [41,51,52] for reviews), one defines, in the case of closed 3-manifolds, the conformal superspace as the space of conformal 3-geometries²⁹, because in this approach gravitational dynamics is regarded as the time evolution of conformal 3-geometry³⁰. See Ref. [10] for the interpretation of the gauge transformations generated by the superhamiltonian constraint: they perform the transition from an allowed 3+1 splitting of spacetime to another one so that the theory is independent from its choice like it happens in parametrized Minkowski theories. Moreover, the Hamiltonian group of gauge transformations of the ADM theory has 8 (and not 4) generators, because, besides the superhamiltonian and supermomentum constraints, there are the four primary first class constraints giving the vanishing of the canonical momenta conjugate to the lapse and shift functions³¹. A preliminary discussion of these problems and of general covariance

²⁵Passive diffeomorphisms are also named *pseudo-diffeomorphisms*.

²⁶It is a stratified manifold with singularities [45].

²⁷In Ref. [8] there is the analogous discussion of the connection of winding number with the great gauge transformations.

²⁸Or better by their induced action on 3-tensors generated by the supermomentum constraints.

²⁹Namely the space of conformal 3-metrics modulo $Diff \Sigma_\tau$ or, equivalently, as $Riem \Sigma_\tau$ (the space of Riemannian 3-metrics) modulo $Diff \Sigma_\tau$ and conformal transformations ${}^3g \mapsto \phi^4 {}^3g$ ($\phi > 0$).

³⁰The momentum conjugate to the conformal factor ϕ is replaced by York time [50,53], i.e. the trace of the extrinsic curvature of Σ_τ .

³¹Whose gauge nature is connected with the gauge nature (conventionality) of simultaneity [54] and of the standards of time and length.

versus Dirac's observables has been given in Ref. [55] ³².

The same happens in tetrad gravity, where there are 14 first class constraints. As we shall see, in our formulation the Hamiltonian gauge group contains: i) a $R^3 \times SO(3)$ subgroup replacing the usual Lorentz subgroup due to our parametrization which Abelianizes Lorentz boosts; ii) $Diff \Sigma_\tau$ in the sense of the pseudo-diffeomorphisms generated by the supermomentum constraints; iii) the gauge transformations generated by a superhamiltonian 1st class constraint; iv) the gauge transformations generated by the momenta conjugate to the lapse and shift functions. In the paper [55] we begun to extract Dirac's observables starting from the symplectic action of infinitesimal diffeomorphisms in $Diff \Sigma_\tau$, ignoring the problems on the structure in large of the component of $Diff \Sigma_\tau$ connected to the identity when a differential structure is posed on it. Although such global properties can be studied in Yang-Mills theory ³³, as shown in Ref. [8], and can be applied to the $SO(3)$ gauge transformations of cotriads ³⁴, one has that $SO(3)$ gauge transformations and $Diff \Sigma_\tau$ do not commute. Therefore, in tetrad gravity the group of $SO(3)$ gauge transformations is an invariant subgroup of a larger group, the group of automorphisms of the $SO(3)$ frame bundle, containing also $Diff \Sigma_\tau$ and again the global situation in the large is of difficult control ³⁵. However, these are topics for future papers.

In Section II the new parametrization of cotetrads is defined.

In Section III such parametrized cotetrads are inserted in the ADM metric action to generate a new Lagrangian for tetrad gravity. The Hamiltonian formulation is developed with the identification of fourteen first class constraints and with the evaluation of their Poisson brackets. The comparison with other formulations of tetrad gravity is done.

In Section IV there is a comparison with ADM canonical metric gravity.

In the Conclusions the next step, namely the identification of the Dirac observables with respect to the gauge transformations generated by thirteen constraints (only the superhamiltonian constraint is not treated) is delineated.

Appendix A is devoted to a review of 4-dimensional pseudo-Riemannian and 3-dimensional Riemannian manifolds asymptotically flat at spatial infinity, of the tetrad and triad formalisms and of the Lagrangians used for general relativity.

Ref. [58] contains an enlarged version of this paper: i) more review material is included in its Section II; ii) there is an Appendix A with the explicit expression of 4-tensors and of the geodesic equation and also with a review on the congruences of timelike worldlines; iii) there is an Appendix B with the Hamiltonian expression of 4-tensors.

³²As also recently noted in Ref. [56] the problem of observables is still open in canonical gravity.

³³Since the group of gauge transformations is a Hilbert-Lie group.

³⁴In our approach the Lorentz boosts are automatically Abelianized.

³⁵ $Diff \Sigma_\tau$ is an inductive limit of Hilbert-Lie groups [57], but the global properties of its group manifold are not well understood.

II. NEW PARAMETRIZATION OF Σ_τ -ADAPTED TETRADS.

As said in the Introduction and in Appendix A, to which we refer for the notations and the definitions, let our globally hyperbolic spacetime M^4 be foliated with spacelike Cauchy hypersurfaces Σ_τ , obtained with the embeddings $i_\tau : \Sigma \rightarrow \Sigma_\tau \subset M^4$, $\vec{\sigma} \mapsto x^\mu = z^\mu(\tau, \vec{\sigma})$, of a 3-manifold Σ in M^4 ³⁶.

In the family of Σ_τ -adapted frames and coframes on M^4 , we can select special tetrads and cotetrads ${}^4_{(\Sigma)}\check{E}_{(\alpha)}$ and ${}^4_{(\Sigma)}\check{\theta}^{(\alpha)}$ also adapted to a given set of triads ${}^3e^r_{(a)}$ and cotriads ${}^3e^{(a)}_r = {}^3e_{(a)r}$ on Σ_τ

$$\begin{aligned} {}^4_{(\Sigma)}\check{E}^\mu_{(\alpha)} &= \{{}^4_{(\Sigma)}\check{E}^\mu_{(o)} = l^\mu = \hat{b}^\mu_l = \frac{1}{N}(b^\mu_\tau - N^r b^\mu_r); {}^4_{(\Sigma)}\check{E}^\mu_{(a)} = {}^3e^s_{(a)} b^\mu_s\}, \\ {}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu &= \{{}^4_{(\Sigma)}\check{E}^{(o)}_\mu = \epsilon l_\mu = \hat{b}^l_\mu = N b^\tau_\mu; {}^4_{(\Sigma)}\check{E}^{(a)}_\mu = {}^3e^{(a)}_s \hat{b}^s_\mu\}, \\ {}^4_{(\Sigma)}\check{E}^\mu_{(\alpha)} {}^4_{g_{\mu\nu}} {}^4_{(\Sigma)}\check{E}^\nu_{(\beta)} &= {}^4\eta_{(\alpha)(\beta)}, \end{aligned} \quad (2.1)$$

where b^μ_r and b^τ_μ are defined in Eqs.(A3) and $l^\mu(\tau, \vec{\sigma})$ is the unit normal to Σ_τ at $\vec{\sigma}$. N and N^r are the standard lapse and shift functions.

The components of these tetrads and cotetrads in the holonomic basis defined in Subsection 1 of Appendix A are respectively (see Refs. [17,24]; ${}^4_{(\Sigma)}\check{E}^{(o)}_r = 0$ is the *Schwinger time gauge condition* [18])

$$\begin{aligned} {}^4_{(\Sigma)}\check{E}^A_{(\alpha)} &= {}^4_{(\Sigma)}\check{E}^\mu_{(\alpha)} b^A_\mu, \quad \Rightarrow {}^4_{(\Sigma)}\check{E}^A_{(o)} = \epsilon l^A, \\ {}^4_{(\Sigma)}\check{E}^\tau_{(o)} &= \frac{1}{N}, \quad {}^4_{(\Sigma)}\check{E}^\tau_{(a)} = 0, \\ {}^4_{(\Sigma)}\check{E}^r_{(o)} &= -\frac{N^r}{N}, \quad {}^4_{(\Sigma)}\check{E}^r_{(a)} = {}^3e^r_{(a)}; \\ {}^4_{(\Sigma)}\check{E}^{(\alpha)}_A &= {}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu b^A_\mu, \quad \Rightarrow {}^4_{(\Sigma)}\check{E}^{(o)}_A = l_A, \\ {}^4_{(\Sigma)}\check{E}^{(o)}_\tau &= N, \quad {}^4_{(\Sigma)}\check{E}^{(a)}_\tau = N^r {}^3e^r_{(a)} = N^{(a)}, \\ {}^4_{(\Sigma)}\check{E}^{(o)}_r &= 0, \quad {}^4_{(\Sigma)}\check{E}^{(a)}_r = {}^3e^{(a)}_r, \\ {}^4_{(\Sigma)}\check{E}^A_{(\alpha)} {}^4_{g_{AB}} {}^4_{(\Sigma)}\check{E}^B_{(\beta)} &= {}^4\eta_{(\alpha)(\beta)}. \end{aligned} \quad (2.2)$$

With the cotetrads ${}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu(z(\sigma))$ we can build the vector $\overset{\circ}{V}^{(\alpha)} = l^\mu(z(\sigma)) {}^4_{(\Sigma)}\check{E}^{(\alpha)}_\mu(z(\sigma)) = (1; \vec{0})$: it is the same unit timelike future-pointing Minkowski 4-vector in the tangent plane of each point $z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma}) \in \Sigma_\tau \subset M^4$ for every τ and $\vec{\sigma}$; we have $\overset{\circ}{V}^{(\alpha)} {}^4_{\eta_{(\alpha)(\beta)}} \overset{\circ}{V}^{(\beta)} = \epsilon$.

³⁶ $\tau : M^4 \rightarrow R$ is a global, timelike, future-oriented function labelling the leaves of the foliation; x^μ are local coordinates in a chart of M^4 ; $\vec{\sigma} = \{\sigma^r\}$, $r=1,2,3$, are local coordinates in a chart of Σ , which is diffeomorphic to R^3 ; we shall use the notation $\sigma^A = (\sigma^\tau = \tau; \vec{\sigma})$, $A = \tau, r$, and $z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma})$.

Let ${}^4E_{(\alpha)}^\mu(z)$ and ${}^4E_\mu^{(\alpha)}(z)$ be arbitrary tetrads and cotetrads on M^4 . Let us define the point-dependent Minkowski 4-vector $V^{(\alpha)}(z(\sigma)) = l^\mu(z(\sigma)) {}^4E_\mu^{(\alpha)}(z(\sigma))$ (assumed to be future-pointing), which satisfies $V^{(\alpha)}(z(\sigma)) {}^4\eta_{(\alpha)(\beta)} V^{(\beta)}(z(\sigma)) = \epsilon$, so that $V^{(\alpha)}(z(\sigma)) = (V^{(o)}(z(\sigma)) = +\sqrt{1 + \sum_r V^{(r)2}(z(\sigma))}; V^{(r)}(z(\sigma)) \stackrel{\text{def}}{=} \varphi^{(r)}(\sigma))$: therefore, the point-dependent Minkowski 4-vector $V^{(\alpha)}(z(\sigma))$ depends only on the three functions $\varphi^{(r)}(\sigma)$ ³⁷. If in each tangent plane we introduce the point-dependent Lorentz transformation

$$\begin{aligned} L^{(\alpha)}_{(\beta)}(V(z(\sigma)); \overset{\circ}{V}) &= \delta^{(\alpha)}_{(\beta)} + 2\epsilon V^{(\alpha)}(z(\sigma)) \overset{\circ}{V}_{(\beta)} - \epsilon \frac{(V^{(\alpha)}(z(\sigma)) + \overset{\circ}{V}^{(\alpha)})(V_{(\beta)}(z(\sigma)) + \overset{\circ}{V}_{(\beta)})}{1 + V^{(o)}(z(\sigma))} = \\ &= \begin{pmatrix} V^{(o)} & -\epsilon V_{(j)} \\ V^{(i)} & \delta^{(i)}_{(j)} - \epsilon \frac{V^{(i)} V_{(j)}}{1 + V^{(o)}} \end{pmatrix} (z(\sigma)), \end{aligned} \quad (2.3)$$

which is the standard Wigner boost for timelike Poincaré orbits [see Ref. [59]], one has by construction

$$V^{(\alpha)}(z(\sigma)) = L^{(\alpha)}_{(\beta)}(V(z(\sigma)); \overset{\circ}{V}) \overset{\circ}{V}^{(\beta)} \stackrel{\text{def}}{=} l^\mu(z(\sigma)) {}^4E_\mu^{(\alpha)}(z(\sigma)). \quad (2.4)$$

We shall define our class of *arbitrary cotetrads* ${}^4E_\mu^{(\alpha)}(z(\sigma))$ on M^4 starting from the special Σ_τ - and cotriad-adapted cotetrads ${}^4_{(\Sigma)}\check{E}_\mu^{(\alpha)}(z(\sigma))$ by means of the formula

$${}^4E_\mu^{(\alpha)}(z(\sigma)) = L^{(\alpha)}_{(\beta)}(V(z(\sigma)); \overset{\circ}{V}) {}^4_{(\Sigma)}\check{E}_\mu^{(\beta)}(z(\sigma)). \quad (2.5)$$

Let us remark that with this definition we are putting equal to zero, by convention, the angles of an arbitrary 3-rotation of $b_\mu^s(z(\sigma))$ ³⁸ inside ${}^4_{(\Sigma)}\check{E}_\mu^{(\alpha)}(z(\sigma))$.

Since $\varphi^{(a)}(\sigma) = V^{(a)}(z(\sigma)) = l^\mu(z(\sigma)) {}^4E_\mu^{(a)}(z(\sigma))$ are the three parameters of the Wigner boost ³⁹, the previous equation can be rewritten in the following form [remembering that $\varphi^{(a)} = -\epsilon\varphi_{(a)}$]

$$\begin{pmatrix} {}^4E_\mu^{(o)} \\ {}^4E_\mu^{(a)} \end{pmatrix} (z(\sigma)) = \begin{pmatrix} \sqrt{1 + \sum_{(c)} \varphi^{(c)2}} & -\epsilon\varphi_{(b)} \\ \varphi^{(a)} & \delta^{(a)}_{(b)} - \epsilon \frac{\varphi^{(a)}\varphi_{(b)}}{1 + \sqrt{1 + \sum_{(c)} \varphi^{(c)2}}} \end{pmatrix} (z(\sigma)) \begin{pmatrix} l_\mu \\ {}^3e_s^{(b)} b_\mu^s \end{pmatrix} (\sigma). \quad (2.6)$$

If we go to holonomic bases, ${}^4E_A^{(\alpha)}(z(\sigma)) = {}^4E_\mu^{(\alpha)}(z(\sigma)) b_A^\mu(\sigma)$ and ${}^4_{(\Sigma)}\check{E}_A^{(\alpha)}(z(\sigma)) = {}^4_{(\Sigma)}\check{E}_\mu^{(\alpha)}(z(\sigma)) b_A^\mu(\sigma)$, one has

³⁷One has $\varphi^{(r)}(\sigma) = -\epsilon\varphi_{(r)}(\sigma)$ since ${}^4\eta_{rs} = -\epsilon\delta_{rs}$; having the Euclidean signature (+++) for both $\epsilon = \pm 1$, we shall define the Kronecker delta as $\delta^{(i)(j)} = \delta^{(i)}_{(j)} = \delta_{(i)(j)}$.

³⁸I.e. of the choice of the three axes tangent to Σ_τ .

³⁹ $\varphi^{(a)} = \bar{\gamma}\beta^{(a)}$, $\bar{\gamma} = \sqrt{1 + \sum_{(c)} \varphi^{(c)2}}$, $\beta^{(a)} = \varphi^{(a)}/\sqrt{1 + \sum_{(c)} \varphi^{(c)2}}$.

$$\begin{pmatrix} {}^4E_A^{(o)} \\ {}^4E_A^{(a)} \end{pmatrix} = \begin{pmatrix} \sqrt{1 + \sum_{(c)} \varphi^{(c)2}} & -\epsilon \varphi^{(b)} \\ \varphi^{(a)} & \delta_{(b)}^{(a)} - \epsilon \frac{\varphi^{(a)} \varphi^{(b)}}{1 + \sqrt{1 + \sum_{(c)} \varphi^{(c)2}}} \end{pmatrix} \times \begin{pmatrix} {}^4_{(\Sigma)} \check{E}_A^{(o)} = (N; \vec{0}) \\ {}^4_{(\Sigma)} \check{E}_A^{(b)} = (N^{(b)} = {}^3e_r^{(b)} N^r; {}^3e_r^{(b)}) \end{pmatrix}, \quad (2.7)$$

so that we get that the cotetrad in the holonomic basis can be expressed in terms of N , $N^{(a)} = {}^3e_s^{(a)} N^s = N_{(a)}$, $\varphi^{(a)}$ and ${}^3e_r^{(a)}$ [${}^3g_{rs} = \sum_{(a)} {}^3e_{(a)r} {}^3e_{(a)s}$]

$$\begin{aligned} {}^4E_\tau^{(o)}(z(\sigma)) &= \sqrt{1 + \sum_{(c)} \varphi^{(c)2}(\sigma)} N(\sigma) + \sum_{(a)} \varphi^{(a)}(\sigma) N^{(a)}(\sigma), \\ {}^4E_r^{(o)}(z(\sigma)) &= \sum_{(a)} \varphi^{(a)}(\sigma) {}^3e_r^{(a)}(\sigma), \\ {}^4E_\tau^{(a)}(z(\sigma)) &= \varphi^{(a)}(\sigma) N(\sigma) + \sum_{(b)} [\delta_{(b)}^{(a)} - \epsilon \frac{\varphi^{(a)}(\sigma) \varphi^{(b)}(\sigma)}{1 + \sqrt{1 + \sum_{(c)} \varphi^{(c)2}(\sigma)}}] N^{(b)}(\sigma), \\ {}^4E_r^{(a)}(z(\sigma)) &= \sum_{(b)} [\delta_{(b)}^{(a)} - \epsilon \frac{\varphi^{(a)}(\sigma) \varphi^{(b)}(\sigma)}{1 + \sqrt{1 + \sum_{(c)} \varphi^{(c)2}(\sigma)}}] {}^3e_r^{(b)}(\sigma), \end{aligned}$$

$$\begin{aligned} \Rightarrow {}^4g_{AB} &= {}^4E_A^{(\alpha)} {}^4\eta_{(\alpha)(\beta)} {}^4E_B^{(\beta)} = {}^4_{(\Sigma)} \check{E}_A^{(\alpha)} {}^4\eta_{(\alpha)(\beta)} {}^4_{(\Sigma)} \check{E}_B^{(\beta)} = \\ &= \epsilon \begin{pmatrix} (N^2 - {}^3g_{rs} N^r N^s) & -{}^3g_{st} N^t \\ -{}^3g_{rt} N^t & -{}^3g_{rs} \end{pmatrix}, \end{aligned} \quad (2.8)$$

with the last line in accord with Eqs.(A3); we have used $L^T {}^4\eta L = {}^4\eta$, valid for every Lorentz transformation. We find $L^{-1}(V, \check{V}) = {}^4\eta L^T(V, \check{V}) {}^4\eta = L(V, \check{V})|_{\varphi^{(a)} \mapsto -\varphi^{(a)}}$ and $[{}^4E_{(\alpha)}^A = {}^4E_{(\alpha)}^\mu b_\mu^A, {}^4_{(\Sigma)} \check{E}_{(\alpha)}^A = {}^4_{(\Sigma)} \check{E}_{(\alpha)}^\mu b_\mu^A]$

$$\begin{aligned} \begin{pmatrix} {}^4E_{(o)}^\mu \\ {}^4E_{(a)}^\mu \end{pmatrix} &= \begin{pmatrix} \sqrt{1 + \sum_{(c)} \varphi^{(c)2}} & -\varphi^{(b)} \\ \epsilon \varphi^{(a)} & \delta_{(a)}^{(b)} - \epsilon \frac{\varphi^{(a)} \varphi^{(b)}}{1 + \sqrt{1 + \sum_{(c)} \varphi^{(c)2}}} \end{pmatrix} \begin{pmatrix} l^\mu \\ b_s^\mu {}^3e_s^{(b)} \end{pmatrix}, \\ \begin{pmatrix} {}^4E_{(o)}^A \\ {}^4E_{(a)}^A \end{pmatrix} &= \begin{pmatrix} \sqrt{1 + \sum_{(c)} \varphi^{(c)2}} & -\varphi^{(b)} \\ \epsilon \varphi^{(a)} & \delta_{(a)}^{(b)} - \epsilon \frac{\varphi^{(a)} \varphi^{(b)}}{1 + \sqrt{1 + \sum_{(c)} \varphi^{(c)2}}} \end{pmatrix} \begin{pmatrix} {}^4_{(\Sigma)} \check{E}_{(o)}^A = (1/N; -N^r/N) \\ {}^4_{(\Sigma)} \check{E}_{(b)}^A = (0; {}^3e_r^{(b)}) \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} {}^4E_{(o)}^\tau(z(\sigma)) &= \sqrt{1 + \sum_{(c)} \varphi^{(c)2}(\sigma)} \frac{1}{N(\sigma)}, \\ {}^4E_{(o)}^r(z(\sigma)) &= -\sqrt{1 + \sum_{(c)} \varphi^{(c)2}(\sigma)} \frac{N^r(\sigma)}{N(\sigma)} - \varphi^{(b)}(\sigma) {}^3e_r^{(b)}(\sigma), \\ {}^4E_{(a)}^\tau(z(\sigma)) &= \epsilon \frac{\varphi^{(a)}(\sigma)}{N(\sigma)}, \end{aligned}$$

$$\begin{aligned}
{}^4E_{(a)}^r(z(\sigma)) &= -\epsilon\varphi_{(a)}(\sigma)\frac{N^r(\sigma)}{N(\sigma)} + \sum_{(b)}[\delta_{(a)}^{(b)} - \epsilon\frac{\varphi_{(a)}(\sigma)\varphi^{(b)}(\sigma)}{1 + \sqrt{1 + \sum_{(c)}\varphi^{(c)2}(\sigma)}}]{}^3e_{(b)}^r(\sigma), \\
\Rightarrow {}^4g^{AB} &= {}^4E_{(\alpha)}^A {}^4\eta^{(\alpha)(\beta)} {}^4E_{(\beta)}^B = {}^4_{(\Sigma)}\check{E}_{(\alpha)}^A {}^4\eta_{(\alpha)(\beta)} {}^4_{(\Sigma)}\check{E}_{(\beta)}^B = \\
&= \epsilon \begin{pmatrix} \frac{1}{N^2} & -\frac{N^s}{N^2} \\ -\frac{N^r}{N^2} & -(^3g^{rs} - \frac{N^rN^s}{N^2}) \end{pmatrix}, \tag{2.9}
\end{aligned}$$

with the last line in accord with Eqs.(A3).

From ${}^4_{(\Sigma)}\check{E}_A^{(\alpha)}(z(\sigma)) = (L^{-1})^{(\alpha)}_{(\beta)}(V(z(\sigma)); \overset{\circ}{V}) {}^4E_A^{(\beta)}(z(\sigma))$ and ${}^4_{(\Sigma)}\check{E}_{(\alpha)}^A(z(\sigma)) = {}^4E_{(\beta)}^A(L^{-1})^{(\beta)}_{(\alpha)}(V(z(\sigma)); \overset{\circ}{V})$ it turns out [59] that the flat indices (a) of the adapted tetrads ${}^4_{(\Sigma)}\check{E}_{(a)}^\mu$ and of the triads ${}^3e_{(a)}^r$ and cotriads ${}^3e_r^{(a)}$ on Σ_τ transform as Wigner spin 1 indices under point-dependent $\text{SO}(3)$ Wigner rotations $R^{(a)}_{(b)}(V(z(\sigma)); \Lambda(z(\sigma)))$ associated with Lorentz transformations $\Lambda^{(\alpha)}_{(\beta)}(z)$ in the tangent plane to M^4 in the same point ⁴⁰. Instead the index (o) of the adapted tetrads ${}^4_{(\Sigma)}\check{E}_{(o)}^\mu$ is a local Lorentz scalar in each point. Therefore, the adapted tetrads in the holonomic basis should be denoted as ${}^4_{(\Sigma)}\check{E}_{(\bar{a})}^A$, with (\bar{o}) and $A = (\tau, r)$ Lorentz scalar indices and with (\bar{a}) Wigner spin 1 indices; we shall go on with the indices $(o), (a)$ without the overbar for the sake of simplicity. In this way the tangent planes to Σ_τ in M^4 are described in a Wigner covariant way, reminiscent of the flat rest-frame covariant instant form of dynamics introduced in Minkowski spacetime in Ref. [7]. Similar conclusions are reached independently in Ref. [60] in the framework of non-linear Poincaré gauge theory ⁴¹.

Therefore, an arbitrary tetrad field, namely a (in general non-geodesic) congruence of observers' timelike worldlines with 4-velocity field $u^A(\tau, \vec{\sigma}) = {}^4E_{(o)}^A(\tau, \vec{\sigma})$, can be obtained with a pointwise Wigner boost from the special surface-forming timelike congruence whose 4-velocity field is the normal to Σ_τ $l^A(\tau, \vec{\sigma}) = \epsilon {}^4_{(\Sigma)}\check{E}_{(o)}^A(\tau, \vec{\sigma})$ ⁴².

We can invert Eqs.(2.9) to get N , $N^r = {}^3e_{(a)}^r N^{(a)}$, $\varphi^{(a)}$ and ${}^3e_{(a)}^r$ in terms of the tetrads ${}^4E_{(\alpha)}^A$

$$\begin{aligned}
N &= \frac{1}{\sqrt{[{}^4E_{(o)}^\tau]^2 - \sum_{(c)}[{}^4E_{(c)}^\tau]^2}}. \\
N^r &= -\frac{{}^4E_{(o)}^\tau {}^4E_{(0)}^r - \sum_{(c)} {}^4E_{(c)}^\tau {}^4E_{(c)}^r}{[{}^4E_{(0)}^\tau]^2 - \sum_{(c)}[{}^4E_{(c)}^\tau]^2}
\end{aligned}$$

$$\begin{aligned}
{}^{40}R^{(\alpha)}_{(\beta)}(\Lambda(z(\sigma)); V(z(\sigma))) &= [L(\overset{\circ}{V}; V(z(\sigma))) \Lambda^{-1}(z(\sigma)) L(\Lambda(z(\sigma))V(z(\sigma)); \overset{\circ}{V})]^{(\alpha)}_{(\beta)} = \\
&\begin{pmatrix} 1 & 0 \\ 0 & R^{(a)}_{(b)}(V(z(\sigma)); \Lambda(z(\sigma))) \end{pmatrix}.
\end{aligned}$$

⁴¹The vector fields e_α and the 1-forms θ^α of that paper correspond to $X_{\bar{A}}$ and $\theta^{\bar{A}}$ in Eq.(A5) respectively.

⁴²It is associated with the 3+1 splitting of M^4 with leaves Σ_τ .

$$\begin{aligned}
\varphi_{(a)} &= \frac{\epsilon \, {}^4E_{(a)}^\tau}{\sqrt{[{}^4E_{(o)}^\tau]^2 - \sum_{(c)} [{}^4E_{(c)}^\tau]^2}} \\
{}^3e_{(a)}^r &= \sum_{(b)} B_{(a)(b)} ({}^4E_{(b)}^r + N^r \, {}^4E_{(b)}^\tau) \\
B_{(a)(b)} &= \delta_{(a)(b)} - \frac{{}^4E_{(a)}^\tau {}^4E_{(b)}^\tau}{{}^4E_{(0)}^\tau [{}^4E_{(0)}^\tau + \sqrt{[{}^4E_{(0)}^\tau]^2 - \sum_{(c)} [{}^4E_{(c)}^\tau]^2}}. \tag{2.10}
\end{aligned}$$

If ${}^3e^{-1} = \det({}^3e_{(a)}^r)$, then from the orthonormality condition we get ${}^3e_{(a)r} = {}^3e({}^3e_{(b)}^s \, {}^3e_{(c)}^t - {}^3e_{(b)}^t \, {}^3e_{(c)}^s)$ ⁴³ and it allows to express the cotriads in terms of the tetrads ${}^4E_{(\alpha)}^A$. Therefore, given the tetrads ${}^4E_{(\alpha)}^A$ (or equivalently the cotetrads ${}^4E_A^{(\alpha)}$) on M^4 , an equivalent set of variables with the local Lorentz covariance replaced with local Wigner covariance are the lapse N , the shifts $N^{(a)} = N_{(a)} = {}^3e_{(a)r} N^r$, the Wigner-boost parameters $\varphi^{(a)} = -\epsilon \varphi_{(a)}$ and either the triads ${}^3e_{(a)}^r$ or the cotriads ${}^3e_{(a)r}$.

⁴³With $(a), (b), (c)$ and r, s, t cyclic.

III. THE LAGRANGIAN AND THE HAMILTONIAN IN THE NEW VARIABLES.

A. The Lagrangian Formulation.

As said in Subsection 4 of Appendix A, we can get an action principle for tetrad gravity starting from the ADM action S_{ADM} (A29):

$$\begin{aligned} S_{ADM} &= -\epsilon \frac{c^3}{16\pi G} \int_U d^4x \sqrt{^4g} [^3R + ^3K_{\mu\nu} ^3K^{\mu\nu} - (^3K)^2] = \\ &= -\epsilon k \int_{\Delta\tau} d\tau \int d^3\sigma \{ \sqrt{\gamma} N [^3R + ^3K_{rs} ^3K^{rs} - (^3K)^2] \} (\tau, \vec{\sigma}). \end{aligned} \quad (3.1)$$

Its independent variables in metric gravity have now the following expression in terms of N , $N^{(a)} = N_{(a)} = ^3e_{(a)}^r N_r$, $\varphi^{(a)} = -\epsilon \varphi_{(a)}$, $^3e_r^{(a)} = ^3e_{(a)r}$ ⁴⁴

$$\begin{aligned} N, \quad N_r &= ^3e_r^{(a)} N_{(a)} = ^3e_{(a)r} N_{(a)}, \\ ^3g_{rs} &= ^3e_r^{(a)} \delta_{(a)(b)} ^3e_s^{(b)} = ^3e_{(a)r} ^3e_{(a)s}, \end{aligned} \quad (3.2)$$

so that the line element of M^4 becomes

$$\begin{aligned} ds^2 &= \epsilon (N^2 - N_{(a)} N_{(a)}) (d\tau)^2 - 2\epsilon N_{(a)} ^3e_{(a)r} d\tau d\sigma^r - \epsilon ^3e_{(a)r} ^3e_{(a)s} d\sigma^r d\sigma^s = \\ &= \epsilon \left[N^2 (d\tau)^2 - (^3e_{(a)r} d\sigma^r + N_{(a)} d\tau) (^3e_{(a)s} d\sigma^s + N_{(a)} d\tau) \right]. \end{aligned} \quad (3.3)$$

The extrinsic curvature takes the form ⁴⁵

$$\begin{aligned} ^3K_{rs} &= \hat{b}_r^\mu \hat{b}_s^\nu ^3K_{\mu\nu} = \frac{1}{2N} (N_{r|s} + N_{s|r} - \partial_\tau ^3g_{rs}) = \\ &= \frac{1}{2N} (^3e_{(a)r} \delta_s^w + ^3e_{(a)s} \delta_r^w) (N_{(a)|w} - \partial_\tau ^3e_{(a)w}), \\ ^3K_{r(a)} &= ^3K_{rs} ^3e_{(a)}^s = \frac{1}{2N} (\delta_{(a)(b)} \delta_r^w + ^3e_{(a)}^w ^3e_{(b)r}) (N_{(b)|w} - \partial_\tau ^3e_{(b)w}), \\ ^3K &= \frac{1}{N} ^3e_{(a)}^r (N_{(a)|r} - \partial_\tau ^3e_{(a)r}), \end{aligned} \quad (3.4)$$

so that the ADM action in the new variables is (from now on we shall use the notation $k = \frac{c^3}{16\pi G}$)

$$\begin{aligned} \hat{S}_{ADMT} &= \int d\tau \hat{L}_{ADMT} = \\ &= -\epsilon k \int d\tau d^3\sigma \{ N ^3e_{(a)(b)(c)} ^3e_{(a)}^r ^3e_{(b)}^s ^3\Omega_{rs(c)} + \\ &+ \frac{^3e}{2N} (^3G_o^{-1})_{(a)(b)(c)(d)} ^3e_{(b)}^r (N_{(a)|r} - \partial_\tau ^3e_{(a)r}) ^3e_{(d)}^s (N_{(c)|s} - \partial_\tau ^3e_{(c)s}) \}, \end{aligned} \quad (3.5)$$

⁴⁴ $\gamma = \det(^3g_{rs}) = (^3e)^2 = (\det(e_{(a)r}))^2$.

⁴⁵ $N_{(a)|r} = ^3e_{(a)}^s N_{s|r} = \partial_r N_{(a)} - \epsilon_{(a)(b)(c)} ^3\omega_{r(b)} N_{(c)}$ from Eq.(A22).

where we introduced the flat inverse Wheeler-DeWitt supermetric

$$({}^3G_o^{-1})_{(a)(b)(c)(d)} = \delta_{(a)(c)}\delta_{(b)(d)} + \delta_{(a)(d)}\delta_{(b)(c)} - 2\delta_{(a)(b)}\delta_{(c)(d)}. \quad (3.6)$$

The flat supermetric is

$$\begin{aligned} {}^3G_{o(a)(b)(c)(d)} &= {}^3G_{o(b)(a)(c)(d)} = {}^3G_{o(a)(b)(d)(c)} = {}^3G_{o(c)(d)(a)(b)} = \\ &= \delta_{(a)(c)}\delta_{(b)(d)} + \delta_{(a)(d)}\delta_{(b)(c)} - \delta_{(a)(b)}\delta_{(c)(d)}, \\ \frac{1}{2} {}^3G_{o(a)(b)(e)(f)} \frac{1}{2} {}^3G_{o(e)(f)(c)(d)}^{-1} &= \frac{1}{2} [\delta_{(a)(c)}\delta_{(b)(d)} + \delta_{(a)(d)}\delta_{(b)(c)}]. \end{aligned} \quad (3.7)$$

The new action does not depend on the 3 boost variables $\varphi^{(a)}$ ⁴⁶, contains lapse N and modified shifts $N_{(a)}$ as Lagrange multipliers, and is a functional independent from the second time derivatives of the fields.

Instead of deriving its Euler-Lagrange equations we shall study its Hamiltonian formulation.

B. The Hamiltonian Formulation.

The canonical momenta and the Poisson brackets are

$$\begin{aligned} \tilde{\pi}_{(a)}^{\vec{\varphi}}(\tau, \vec{\sigma}) &= \frac{\delta \hat{S}_{ADMT}}{\delta \partial_\tau \varphi_{(a)}(\tau, \vec{\sigma})} = 0, \\ \tilde{\pi}^N(\tau, \vec{\sigma}) &= \frac{\delta \hat{S}_{ADMT}}{\delta \partial_\tau N(\tau, \vec{\sigma})} = 0, \\ \tilde{\pi}_{(a)}^{\vec{N}}(\tau, \vec{\sigma}) &= \frac{\delta \hat{S}_{ADMT}}{\delta \partial_\tau N_{(a)}(\tau, \vec{\sigma})} = 0, \\ {}^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}) &= \frac{\delta \hat{S}_{ADMT}}{\delta \partial_\tau {}^3e_{(a)r}(\tau, \vec{\sigma})} = [\frac{\epsilon k^3 e}{N} ({}^3G_o^{-1})_{(a)(b)(c)(d)} {}^3e_{(b)}^r {}^3e_{(c)}^s (N_{(c)|s} - \partial_\tau {}^3e_{(c)s})](\tau, \vec{\sigma}) = \\ &= 2\epsilon k [{}^3e({}^3K^{rs} - {}^3e_{(c)}^r {}^3e_{(c)}^s {}^3K) {}^3e_{(a)s}](\tau, \vec{\sigma}), \\ \{N(\tau, \vec{\sigma}), \tilde{\pi}^N(\tau, \vec{\sigma}')\} &= \delta^3(\vec{\sigma}, \vec{\sigma}'), \\ \{N_{(a)}(\tau, \vec{\sigma}), \tilde{\pi}_{(b)}^{\vec{N}}(\tau, \vec{\sigma}')\} &= \delta_{(a)(b)} \delta^3(\vec{\sigma}, \vec{\sigma}'), \\ \{\varphi_{(a)}(\tau, \vec{\sigma}), \tilde{\pi}_{(b)}^{\vec{\varphi}}(\tau, \vec{\sigma}')\} &= \delta_{(a)(b)} \delta^3(\vec{\sigma}, \vec{\sigma}'), \\ \{{}^3e_{(a)r}(\tau, \vec{\sigma}), {}^3\tilde{\pi}_{(b)}^s(\tau, \vec{\sigma}')\} &= \delta_{(a)(b)} \delta_r^s \delta^3(\vec{\sigma}, \vec{\sigma}'), \\ \{{}^3e_{(a)}^r(\tau, \vec{\sigma}), {}^3\tilde{\pi}_{(b)}^s(\tau, \vec{\sigma}')\} &= -{}^3e_{(b)}^r(\tau, \vec{\sigma}) {}^3e_{(a)}^s(\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}'), \\ \{{}^3e(\tau, \vec{\sigma}), {}^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}')\} &= {}^3e(\tau, \vec{\sigma}) {}^3e_{(a)}^r(\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}'), \end{aligned} \quad (3.8)$$

⁴⁶Like the Higgs model Lagrangian in the unitary gauge does not depend on some of the Higgs fields [61,62].

where the Dirac delta distribution is a density of weight -1⁴⁷. The momentum ${}^3\tilde{\pi}_{(a)}^r$ is a density of weight -1.

Besides the seven primary constraints

$$\begin{aligned}\tilde{\pi}_{(a)}^{\vec{\sigma}}(\tau, \vec{\sigma}) &\approx 0, \\ \tilde{\pi}^N(\tau, \vec{\sigma}) &\approx 0, \\ \tilde{\pi}_{(a)}^{\vec{N}}(\tau, \vec{\sigma}) &\approx 0,\end{aligned}\tag{3.9}$$

there are the following three primary constraints (the generators of the inner rotations)

$$\begin{aligned}{}^3\tilde{M}_{(a)}(\tau, \vec{\sigma}) &= \epsilon_{(a)(b)(c)} {}^3e_{(b)r}(\tau, \vec{\sigma}) {}^3\tilde{\pi}_{(c)}^r(\tau, \vec{\sigma}) = \frac{1}{2}\epsilon_{(a)(b)(c)} {}^3\tilde{M}_{(b)(c)}(\tau, \vec{\sigma}) \approx 0, \\ \Rightarrow {}^3\tilde{M}_{(a)(b)}(\tau, \vec{\sigma}) &= \epsilon_{(a)(b)(c)} {}^3\tilde{M}_{(c)}(\tau, \vec{\sigma}) = \\ &= {}^3e_{(a)r}(\tau, \vec{\sigma}) {}^3\tilde{\pi}_{(b)}^r(\tau, \vec{\sigma}) - {}^3e_{(b)r}(\tau, \vec{\sigma}) {}^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}) \approx 0.\end{aligned}\tag{3.10}$$

By using Eqs.(3.7) and (3.8) we get the following inversion

$$\begin{aligned}{}^3e_{(a)}^r (N_{(b)|r} - \partial_\tau {}^3e_{(b)r}) + {}^3e_{(b)}^r (N_{(a)|r} - \partial_\tau {}^3e_{(a)r}) &= \\ = \frac{\epsilon N}{2k {}^3e} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(c)r} {}^3\tilde{\pi}_{(d)}^r,\end{aligned}\tag{3.11}$$

so that, even if, due to the degeneracy associated with the first class constraints, this equation cannot be solved for $\partial_\tau {}^3e_{(a)r}$, we can get the phase space expression of the extrinsic curvature without using the Hamilton equations

$$\begin{aligned}{}^3K_{rs} &= \frac{\epsilon}{4k {}^3e} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(a)r} {}^3e_{(b)s} {}^3e_{(c)u} {}^3\tilde{\pi}_{(d)}^u, \\ {}^3K &= -\frac{\epsilon}{2k\sqrt{\gamma}} {}^3\tilde{\Pi} = -\frac{\epsilon}{4k {}^3e} {}^3e_{(a)r} {}^3\tilde{\pi}_{(a)}^r.\end{aligned}\tag{3.12}$$

Since at the Lagrangian level the primary constraints are identically zero, we have

$$\begin{aligned}{}^3\tilde{\pi}_{(a)}^r &= {}^3e_{(b)}^r {}^3e_{(b)s} {}^3\tilde{\pi}_{(a)}^s = \frac{1}{2} {}^3e_{(b)}^r [{}^3e_{(a)s} {}^3\tilde{\pi}_{(b)}^s + {}^3e_{(b)s} {}^3\tilde{\pi}_{(a)}^s] - \frac{1}{2} {}^3\tilde{M}_{(a)(b)} {}^3e_{(b)}^r \equiv \\ &\equiv \frac{1}{2} {}^3e_{(b)}^r [{}^3e_{(a)s} {}^3\tilde{\pi}_{(b)}^s + {}^3e_{(b)s} {}^3\tilde{\pi}_{(a)}^s], \\ {}^3\tilde{\pi}_{(a)}^r \partial_\tau {}^3e_{(a)r} &\equiv \frac{1}{2} [{}^3e_{(a)s} {}^3\tilde{\pi}_{(b)}^s + {}^3e_{(b)s} {}^3\tilde{\pi}_{(a)}^s] {}^3e_{(b)}^r \partial_\tau {}^3e_{(a)r} \equiv \\ &\equiv {}^3\tilde{\pi}_{(a)}^r N_{(a)|r} - \frac{N}{4k {}^3e} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(a)s} {}^3\tilde{\pi}_{(b)}^s {}^3e_{(c)r} {}^3\tilde{\pi}_{(d)}^r,\end{aligned}\tag{3.13}$$

and the canonical Hamiltonian is

⁴⁷It behaves as $\sqrt{\gamma(\tau, \vec{\sigma})}$, because we have the $\vec{\sigma}'$ -reparametrization invariant result $\int d^3\sigma' \delta^3(\vec{\sigma}, \vec{\sigma}') f(\vec{\sigma}') = f(\vec{\sigma})$.

$$\begin{aligned}
\hat{H}_{(c)} &= \int d^3\sigma [\tilde{\pi}^N \partial_\tau N + \tilde{\pi}_{(a)}^{\vec{N}} \partial_\tau N_{(a)} + \tilde{\pi}_{(a)}^{\vec{\varphi}} \partial_\tau \varphi_{(a)} + {}^3\tilde{\pi}_{(a)}^r \partial_\tau {}^3e_{(a)r}] (\tau, \vec{\sigma}) - \hat{L}_{ADMT} = \\
&= \int_{\Sigma_\tau} d^3\sigma [\epsilon N (k {}^3e \epsilon_{(a)(b)(c)} {}^3e_{(a)}^r {}^3e_{(b)}^s {}^3\Omega_{rs(c)} - \\
&\quad - \frac{1}{8k {}^3e} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(a)r} {}^3\tilde{\pi}_{(b)}^r {}^3e_{(c)s} {}^3\tilde{\pi}_{(d)}^s) - \\
&\quad - N_{(a)} {}^3\tilde{\pi}_{(a)|r}^r] (\tau, \vec{\sigma}) + \int_{\partial\Sigma_\tau} d^2\Sigma_r [N_{(a)} {}^3\tilde{\pi}_{(a)}^r] (\tau, \vec{\sigma}). \tag{3.14}
\end{aligned}$$

In this paper we shall ignore the surface term.

The Dirac Hamiltonian is (the $\lambda(\tau, \vec{\sigma})$'s are arbitrary Dirac multipliers)

$$\hat{H}_{(D)} = \hat{H}_{(c)} + \int d^3\sigma [\lambda_N \tilde{\pi}^N + \lambda_{(a)}^{\vec{N}} \tilde{\pi}_{(a)}^{\vec{N}} + \lambda_{(a)}^{\vec{\varphi}} \tilde{\pi}_{(a)}^{\vec{\varphi}} + \mu_{(a)} {}^3\tilde{M}_{(a)}] (\tau, \vec{\sigma}). \tag{3.15}$$

The τ -constancy of the ten primary constraints ($\partial_\tau \tilde{\pi}^N(\tau, \vec{\sigma}) \approx 0$ and $\partial_\tau \tilde{\pi}_{(a)}^{\vec{N}}(\tau, \vec{\sigma}) \approx 0$) generates four secondary constraints

$$\begin{aligned}
\hat{\mathcal{H}}(\tau, \vec{\sigma}) &= \epsilon [k {}^3e \epsilon_{(a)(b)(c)} {}^3e_{(a)}^r {}^3e_{(b)}^s {}^3\Omega_{rs(c)} - \\
&\quad - \frac{1}{8k {}^3e} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(a)r} {}^3\tilde{\pi}_{(b)}^r {}^3e_{(c)s} {}^3\tilde{\pi}_{(d)}^s] (\tau, \vec{\sigma}) = \\
&= \epsilon [k {}^3e {}^3R - \frac{1}{8k {}^3e} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(a)r} {}^3\tilde{\pi}_{(b)}^r {}^3e_{(c)s} {}^3\tilde{\pi}_{(d)}^s] (\tau, \vec{\sigma}) \approx 0, \\
\hat{\mathcal{H}}_{(a)}(\tau, \vec{\sigma}) &= [\partial_r {}^3\tilde{\pi}_{(a)}^r - \epsilon_{(a)(b)(c)} {}^3\omega_{r(b)} {}^3\tilde{\pi}_{(c)}^r] (\tau, \vec{\sigma}) = {}^3\tilde{\pi}_{(a)|r}^r(\tau, \vec{\sigma}) \approx 0, \\
\Rightarrow \hat{H}_{(c)} &= \int d^3\sigma [N \hat{\mathcal{H}} - N_{(a)} \hat{\mathcal{H}}_{(a)}] (\tau, \vec{\sigma}) \approx 0. \tag{3.16}
\end{aligned}$$

It can be checked that the superhamiltonian constraint $\hat{\mathcal{H}}(\tau, \vec{\sigma}) \approx 0$ coincides with the ADM metric superhamiltonian one $\tilde{\mathcal{H}}(\tau, \vec{\sigma}) \approx 0$ given in Eqs.(4.10) of Section IV, where also the ADM metric supermomentum constraints is expressed in terms of the tetrad gravity constraints.

It is convenient to replace the constraints $\hat{\mathcal{H}}_{(a)}(\tau, \vec{\sigma}) \approx 0$ ⁴⁸ with the 3 constraints generating space pseudo-diffeomorphisms on the cotriads and their conjugate momenta

$$\begin{aligned}
{}^3\tilde{\Theta}_r(\tau, \vec{\sigma}) &= -[{}^3e_{(a)r} \hat{\mathcal{H}}_{(a)} + {}^3\omega_{r(a)} {}^3\tilde{M}_{(a)}] (\tau, \vec{\sigma}) = \\
&= [{}^3\tilde{\pi}_{(a)}^s \partial_r {}^3e_{(a)s} - \partial_s ({}^3e_{(a)r} {}^3\tilde{\pi}_{(a)}^s)] (\tau, \vec{\sigma}) \approx 0,
\end{aligned}$$

$$\hat{\mathcal{H}}_{(a)}(\tau, \vec{\sigma}) = -{}^3e_{(a)}^r(\tau, \vec{\sigma}) [{}^3\tilde{\Theta}_r + {}^3\omega_{r(b)} {}^3\tilde{M}_{(b)}] (\tau, \vec{\sigma}) \approx 0,$$

$$\begin{aligned}
\Rightarrow \hat{H}_{(D)} &= \hat{H}'_{(c)} + \int d^3\sigma [\lambda_N \tilde{\pi}^N + \lambda_{(a)}^{\vec{N}} \tilde{\pi}_{(a)}^{\vec{N}} + \lambda_{(a)}^{\vec{\varphi}} \tilde{\pi}_{(a)}^{\vec{\varphi}} + \hat{\mu}_{(a)} {}^3\tilde{M}_{(a)}] (\tau, \vec{\sigma}), \\
\hat{H}'_{(c)} &= \int d^3\sigma [N \hat{\mathcal{H}} + N_{(a)} {}^3e_{(a)}^r {}^3\tilde{\Theta}_r] (\tau, \vec{\sigma}), \tag{3.17}
\end{aligned}$$

⁴⁸They are of the type of SO(3) Yang-Mills Gauss laws, because they are the covariant divergence of a vector density.

where we replaced $[\mu_{(a)} - N_{(b)} {}^3e_{(b)}^r {}^3\omega_{r(a)}](\tau, \vec{\sigma})$ with the new Dirac multipliers $\hat{\mu}_{(a)}(\tau, \vec{\sigma})$.

All the constraints are first class because the only non-identically vanishing Poisson brackets are

$$\begin{aligned}
\{ {}^3\tilde{M}_{(a)}(\tau, \vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau, \vec{\sigma}') \} &= \epsilon_{(a)(b)(c)} {}^3\tilde{M}_{(c)}(\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\{ {}^3\tilde{M}_{(a)}(\tau, \vec{\sigma}), {}^3\tilde{\Theta}_r(\tau, \vec{\sigma}') \} &= {}^3\tilde{M}_{(a)}(\tau, \vec{\sigma}') \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^r}, \\
\{ {}^3\tilde{\Theta}_r(\tau, \vec{\sigma}), {}^3\tilde{\Theta}_s(\tau, \vec{\sigma}') \} &= [{}^3\tilde{\Theta}_r(\tau, \vec{\sigma}') \frac{\partial}{\partial \sigma^s} + {}^3\tilde{\Theta}_s(\tau, \vec{\sigma}) \frac{\partial}{\partial \sigma^r}] \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\{ \hat{\mathcal{H}}(\tau, \vec{\sigma}), {}^3\tilde{\Theta}_r(\tau, \vec{\sigma}') \} &= \hat{\mathcal{H}}(\tau, \vec{\sigma}') \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^r}, \\
\{ \hat{\mathcal{H}}(\tau, \vec{\sigma}), \hat{\mathcal{H}}(\tau, \vec{\sigma}') \} &= [{}^3e_{(a)}^r(\tau, \vec{\sigma}) \hat{\mathcal{H}}_{(a)}(\tau, \vec{\sigma}) + \\
&\quad + {}^3e_{(a)}^r(\tau, \vec{\sigma}') \hat{\mathcal{H}}_{(a)}(\tau, \vec{\sigma}')] \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^r} = \\
&= \{ [{}^3e_{(a)}^r {}^3e_{(a)}^s [{}^3\tilde{\Theta}_s + {}^3\omega_{s(b)} {}^3\tilde{M}_{(b)}]](\tau, \vec{\sigma}) + \\
&\quad + [{}^3e_{(a)}^r {}^3e_{(a)}^s [{}^3\tilde{\Theta}_s + {}^3\omega_{s(b)} {}^3\tilde{M}_{(b)}]](\tau, \vec{\sigma}') \} \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^r}. \tag{3.18}
\end{aligned}$$

As said at the end of the Introduction, the Hamiltonian gauge group has the 14 first class constraints as generators of infinitesimal gauge transformations connected with the identity. In particular $\tilde{\pi}_{(a)}^{\vec{\sigma}}(\tau, \vec{\sigma}) \approx 0$ and ${}^3\tilde{M}_{(a)}(\tau, \vec{\sigma}) \approx 0$ are the generators of the $R^3 \times SO(3)$ subgroup replacing the Lorentz subgroup with our parametrization, while ${}^3\tilde{\Theta}_r(\tau, \vec{\sigma}) \approx 0$ are the generators of the infinitesimal pseudodiffeomorphisms in $Diff \Sigma_\tau$.

The Poisson brackets of the cotriads and of their conjugate momenta with the constraints are $[{}^3R = \epsilon_{(a)(b)(c)} {}^3e_{(a)}^r {}^3e_{(b)}^s {}^3\Omega_{rs(c)}]$

$$\begin{aligned}
\{ {}^3e_{(a)r}(\tau, \vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau, \vec{\sigma}') \} &= \epsilon_{(a)(b)(c)} {}^3e_{(c)r}(\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\{ {}^3e_{(a)r}(\tau, \vec{\sigma}), {}^3\tilde{\Theta}_s(\tau, \vec{\sigma}') \} &= \frac{\partial {}^3e_{(a)r}(\tau, \vec{\sigma})}{\partial \sigma^s} \delta^3(\vec{\sigma}, \vec{\sigma}') + {}^3e_{(a)s}(\tau, \vec{\sigma}) \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^r}, \\
\{ {}^3e_{(a)r}(\tau, \vec{\sigma}), \hat{\mathcal{H}}(\tau, \vec{\sigma}') \} &= -\frac{\epsilon}{4k} \left[\frac{1}{3e} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(b)r} {}^3e_{(c)s} {}^3\tilde{\pi}_{(d)}^s \right] (\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\{ {}^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}), {}^3\tilde{M}_{(b)}(\tau, \vec{\sigma}') \} &= \epsilon_{(a)(b)(c)} {}^3\tilde{\pi}_{(c)}^r(\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\{ {}^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}), {}^3\tilde{\Theta}_s(\tau, \vec{\sigma}') \} &= -{}^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}') \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma'^s} + \delta_s^r \frac{\partial}{\partial \sigma'^u} [{}^3\tilde{\pi}_{(a)}^u(\tau, \vec{\sigma}') \delta^3(\vec{\sigma}, \vec{\sigma}')], \\
\{ {}^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}), \hat{\mathcal{H}}(\tau, \vec{\sigma}') \} &= \epsilon \left[2k {}^3e ({}^3R^{rs} - \frac{1}{2} {}^3g^{rs} {}^3R) {}^3e_{(a)s} + \right. \\
&\quad + \frac{1}{4k} {}^3G_{o(a)(b)(c)(d)} {}^3\tilde{\pi}_{(b)}^r {}^3e_{(c)s} {}^3\tilde{\pi}_{(d)}^s - \\
&\quad - \frac{1}{8k} {}^3e_{(a)}^r {}^3G_{o(b)(c)(d)(e)} {}^3e_{(b)u} {}^3\tilde{\pi}_{(c)}^u {}^3e_{(d)v} {}^3\tilde{\pi}_{(e)}^v \left. \right] (\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}') + \\
&\quad + 2k {}^3e(\tau, \vec{\sigma}) \left[{}^3\Gamma_{uv}^w ({}^3e_{(a)}^u {}^3g^{rv} - {}^3e_{(a)}^r {}^3g^{uv}) \right] (\tau, \vec{\sigma}') \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^w} + \\
&\quad + 2k {}^3e(\tau, \vec{\sigma}) \left[{}^3e_{(a)}^u {}^3g^{rv} - {}^3e_{(a)}^r {}^3g^{uv} \right] (\tau, \vec{\sigma}') \frac{\partial^2 \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^u \partial \sigma^v}, \tag{3.19}
\end{aligned}$$

where we used

$$\begin{aligned} \{^3e(\tau, \vec{\sigma})^3R(\tau, \vec{\sigma}), ^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}')\} &= -2k \left[^3e(^3R^{rs} - \frac{1}{2}^3g^{rs}^3R)^3e_{(a)s} \right] (\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}') + \\ &+ 2k ^3e(\tau, \vec{\sigma}) \left[^3\Gamma_{uv}^{(3}e_{(a)}^u ^3g^{rv} - ^3e_{(a)}^r ^3g^{uv} \right] (\tau, \vec{\sigma}') \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^w} + \\ &+ 2k ^3e(\tau, \vec{\sigma}) \left[^3e_{(a)}^u ^3g^{rv} - ^3e_{(a)}^r ^3g^{uv} \right] (\tau, \vec{\sigma}') \frac{\partial^2 \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^u \partial \sigma^v}. \end{aligned}$$

The Hamilton equations associated with the Dirac Hamiltonian (3.17) are (see Eqs.(A25) for $^3R^{uv}$)

$$\begin{aligned} \partial_\tau N(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{N(\tau, \vec{\sigma}), \hat{H}'_{(D)}\} = \lambda_N(\tau, \vec{\sigma}), \\ \partial_\tau N_{(a)}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{N_{(a)}(\tau, \vec{\sigma}), \hat{H}'_{(D)}\} = \lambda_{(a)}^{\vec{N}}(\tau, \vec{\sigma}), \\ \partial_\tau \varphi_{(a)}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{\varphi_{(a)}(\tau, \vec{\sigma}), \hat{H}'_{(D)}\} = \lambda_{(a)}^{\vec{\varphi}}(\tau, \vec{\sigma}), \\ \partial_\tau ^3e_{(a)r}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{^3e_{(a)r}(\tau, \vec{\sigma}), \hat{H}'_{(D)}\} = \\ &= -\frac{\epsilon}{4k} \left[\frac{N}{^3e} ^3G_{o(a)(b)(c)(d)} ^3e_{(b)r} ^3e_{(c)s} ^3\tilde{\pi}_{(d)}^s \right] (\tau, \vec{\sigma}) + \\ &+ \left[N_{(b)} ^3e_{(b)}^s \frac{\partial ^3e_{(a)r}}{\partial \sigma^s} + ^3e_{(a)s} \frac{\partial}{\partial \sigma^r} (N_{(b)} ^3e_{(b)}^s) \right] (\tau, \vec{\sigma}) + \\ &+ \epsilon_{(a)(b)(c)} \hat{\mu}_{(b)}(\tau, \vec{\sigma}) ^3e_{(c)r}(\tau, \vec{\sigma}), \\ \partial_\tau ^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}), \hat{H}'_{(D)}\} = \\ &= 2k\epsilon \left[^3eN(^3R^{rs} - \frac{1}{2}^3g^{rs}^3R)^3e_{(a)s} + ^3e(N^{|r|s} - ^3g^{rs}N^{|u}_{|u})^3e_{(a)s} \right] (\tau, \vec{\sigma}) - \\ &- \epsilon \frac{N(\tau, \vec{\sigma})}{8k} \left[\frac{1}{^3e} ^3G_{o(a)(b)(c)(d)} ^3\tilde{\pi}_{(b)}^r ^3e_{(c)s} ^3\tilde{\pi}_{(d)}^s - \right. \\ &- \frac{2}{^3e} ^3e_{(a)}^r ^3G_{o(b)(c)(d)(e)} ^3e_{(b)u} ^3\tilde{\pi}_{(c)}^u ^3e_{(d)v} ^3\tilde{\pi}_{(e)}^v \left. \right] (\tau, \vec{\sigma}) + \\ &+ \frac{\partial}{\partial \sigma^s} \left[N_{(b)} ^3e_{(b)}^s ^3\tilde{\pi}_{(a)}^r \right] (\tau, \vec{\sigma}) - ^3\tilde{\pi}_{(a)}^u(\tau, \vec{\sigma}) \frac{\partial}{\partial \sigma^u} \left[N_{(b)} ^3e_{(b)}^r \right] (\tau, \vec{\sigma}) + \\ &+ \epsilon_{(a)(b)(c)} \hat{\mu}_{(b)}(\tau, \vec{\sigma}) ^3\tilde{\pi}_{(c)}^r(\tau, \vec{\sigma}), \\ &\Downarrow \\ \partial_\tau ^3e_{(a)}^r(\tau, \vec{\sigma}) &= - \left[^3e_{(b)}^r ^3e_{(a)}^s \partial_\tau ^3e_{(b)s} \right] (\tau, \vec{\sigma}) \stackrel{\circ}{=} \\ &\stackrel{\circ}{=} \frac{\epsilon}{4k} \left[\frac{N}{^3e} ^3G_{o(a)(b)(c)(d)} ^3e_{(b)}^r ^3e_{(c)s} ^3\tilde{\pi}_{(d)}^s \right] (\tau, \vec{\sigma}) - \\ &- ^3e_{(a)}^s \left[N_{(c)} ^3e_{(c)}^u ^3e_{(b)}^r \frac{\partial ^3e_{(b)s}}{\partial \sigma^u} + \frac{\partial}{\partial \sigma^s} (N_{(c)} ^3e_{(c)}^r) \right] (\tau, \vec{\sigma}) + \\ &+ \epsilon_{(a)(b)(c)} \hat{\mu}_{(b)}(\tau, \vec{\sigma}) ^3e_{(c)}^r(\tau, \vec{\sigma}), \\ \partial_\tau ^3e(\tau, \vec{\sigma}) &= \left[^3e ^3e_{(a)}^r \partial_\tau ^3e_{(a)r} \right] (\tau, \vec{\sigma}) \stackrel{\circ}{=} \\ &\stackrel{\circ}{=} \frac{\epsilon}{4k} \left[N ^3e_{(a)s} ^3\tilde{\pi}_{(a)}^s \right] (\tau, \vec{\sigma}) + \\ &+ \left(^3e \left[N_{(b)} ^3e_{(b)}^s ^3e_{(a)}^r \partial_s ^3e_{(a)r} + ^3e_{(a)}^r ^3e_{(a)s} \partial_r (N_{(b)} ^3e_{(b)}^s) \right] \right) (\tau, \vec{\sigma}). \end{aligned} \tag{3.20}$$

From the Hamilton equations and Eqs.(A25), (3.12), we get

$$\begin{aligned}
\partial_\tau {}^3g_{rs}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} [N_{r|s} + N_{s|r} - 2N {}^3K_{rs}](\tau, \vec{\sigma}), \\
\partial_\tau {}^3K_{rs}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \frac{1}{4k} {}^3G_{o(a)(b)(c)(d)} \left(\frac{\epsilon}{3e} \left[\partial_v (N_{(m)} {}^3e_{(m)}^v {}^3e_{(a)r} {}^3e_{(b)s} {}^3e_{(c)u} {}^3\tilde{\pi}_{(d)}^u) + \right. \right. \\
&\quad + {}^3e_{(c)u} {}^3\tilde{\pi}_{(d)}^u \left[2kN({}^3R^{uv} - \frac{1}{2} {}^3g^{uv} {}^3R) + \epsilon(N^{[u|v} - {}^3g^{uv} N^{l|l]} \right] {}^3e_{(d)v} - \\
&\quad - \frac{N}{4k {}^3e^2} \left[\frac{1}{2} {}^3e_{(a)r} {}^3e_{(b)s} {}^3e_{(c)u} {}^3G_{o(d)(e)(f)(g)} {}^3\tilde{\pi}_{(e)}^u {}^3e_{(f)v} {}^3\tilde{\pi}_{(g)}^v - \right. \\
&\quad - {}^3e_{(a)r} {}^3e_{(b)s} \delta_{(c)(d)} {}^3G_{o(e)(f)(g)(h)} {}^3e_{(e)u} {}^3\tilde{\pi}_{(f)}^u {}^3e_{(g)v} {}^3\tilde{\pi}_{(h)}^v + \\
&\quad + {}^3e_{(a)r} {}^3e_{(b)s} ({}^3e_{(m)v} {}^3\tilde{\pi}_{(m)}^v {}^3e_{(c)u} {}^3\tilde{\pi}_{(d)}^u + {}^3G_{o(c)(e)(f)(g)} {}^3e_{(e)u} {}^3\tilde{\pi}_{(d)}^u {}^3e_{(f)v} {}^3\tilde{\pi}_{(g)}^v) + \\
&\quad \left. \left. + ({}^3e_{(a)r} {}^3G_{o(b)(e)(f)(g)} {}^3e_{(e)s} + {}^3e_{(b)s} {}^3G_{o(a)(e)(f)(g)} {}^3e_{(e)r}) \right. \right. \\
&\quad \left. \left. {}^3e_{(f)u} {}^3\tilde{\pi}_{(g)}^u {}^3e_{(c)v} {}^3\tilde{\pi}_{(d)}^v \right] \right) (\tau, \vec{\sigma}), \\
\partial_\tau {}^3K(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \left(\frac{1}{4} N {}^3R + 4N^{lr}{}_{|r} + \right. \\
&\quad + \frac{N}{(4k {}^3e)^2} \left[({}^3e_{(a)r} {}^3\tilde{\pi}_{(a)}^r)^2 - \frac{3}{2} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(a)r} {}^3\tilde{\pi}_{(b)}^r {}^3e_{(c)s} {}^3\tilde{\pi}_{(d)}^s \right] - \\
&\quad \left. - \frac{\epsilon}{4k {}^3e} {}^3\tilde{\pi}_{(a)}^r \left[N_{(m)} {}^3e_{(m)}^u \partial_u {}^3e_{(a)r} + {}^3e_{(a)u} \partial_r (N_{(m)} {}^3e_{(m)}^u) \right] \right) (\tau, \vec{\sigma}), \\
\partial_\tau {}^3\omega_{r(a)(b)}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \frac{\epsilon N}{4k {}^3e} \left((\partial_r {}^3e_{(b)s} - \partial_s {}^3e_{(b)r}) {}^3G_{o(a)(l)(m)(n)} {}^3e_{(l)}^s + \right. \\
&\quad + (\partial_s {}^3e_{(a)r} - \partial_r {}^3e_{(a)s}) {}^3G_{o(b)(l)(m)(n)} {}^3e_{(l)}^s + \\
&\quad + (\partial_v {}^3e_{(c)u} - \partial_u {}^3e_{(c)v}) \left[{}^3e_{(b)}^v {}^3e_{(c)r} {}^3G_{o(a)(l)(m)(n)} {}^3e_{(l)}^u + \right. \\
&\quad + {}^3e_{(a)}^u {}^3e_{(c)r} {}^3G_{o(b)(l)(m)(n)} {}^3e_{(l)}^v - \\
&\quad \left. \left. - {}^3e_{(a)}^u {}^3e_{(b)}^v {}^3G_{o(c)(l)(m)(n)} {}^3e_{(l)r} \right] {}^3e_{(m)t} {}^3\tilde{\pi}_{(n)}^t \right) - \\
&\quad - \frac{\epsilon}{4k} \left(\left[{}^3e_{(a)}^s {}^3G_{o(b)(l)(m)(n)} - {}^3e_{(b)}^s {}^3G_{o(a)(l)(m)(n)} \right] \right. \\
&\quad \left[\partial_r \left(\frac{N}{3e} {}^3e_{(l)s} {}^3e_{(m)t} {}^3\tilde{\pi}_{(n)}^t \right) - \partial_s \left(\frac{N}{3e} {}^3e_{(l)r} {}^3e_{(m)t} {}^3\tilde{\pi}_{(n)}^t \right) \right] + \\
&\quad + {}^3e_{(a)}^u {}^3e_{(b)}^v {}^3e_{(c)r} {}^3G_{o(c)(l)(m)(n)} \\
&\quad \left[\partial_v \left(\frac{N}{3e} {}^3e_{(l)u} {}^3e_{(m)t} {}^3\tilde{\pi}_{(n)}^t \right) - \partial_u \left(\frac{N}{3e} {}^3e_{(l)v} {}^3e_{(m)t} {}^3\tilde{\pi}_{(n)}^t \right) \right] \Big) - \\
&\quad - \left[(\partial_r {}^3e_{(b)s} - \partial_s {}^3e_{(b)r}) {}^3e_{(a)}^v - (\partial_r {}^3e_{(a)s} - \partial_s {}^3e_{(a)r}) {}^3e_{(b)}^v \right] \\
&\quad \left[N_{(w)} {}^3e_{(w)}^u {}^3e_{(l)}^s \partial_u (N_{(w)} {}^3e_{(w)}^s) \right] - \\
&\quad - (\partial_v {}^3e_{(c)u} - \partial_u {}^3e_{(c)v}) \left[({}^3e_{(b)}^v {}^3e_{(a)}^t + {}^3e_{(a)}^u {}^3e_{(b)}^t) {}^3e_{(c)r} \right. \\
&\quad \left. \left[N_{(m)} {}^3e_{(m)}^w {}^3e_{(l)}^u \partial_w {}^3e_{(l)t} + \partial_t (N_{(w)} {}^3e_{(w)}^u) \right] + \right. \\
&\quad + {}^3e_{(a)}^u {}^3e_{(b)}^v \left(N_{(m)} {}^3e_{(m)}^w \partial_w {}^3e_{(c)r} + {}^3e_{(c)w} \partial_r (N_{(m)} {}^3e_{(m)}^w) \right) \Big] + \\
&\quad + {}^3e_{(a)}^s \left(\partial_r (N_{(w)} {}^3e_{(w)}^u \partial_u {}^3e_{(b)s} + {}^3e_{(b)u} \partial_s (N_{(w)} {}^3e_{(w)}^u)) - \right. \\
&\quad - \partial_s (N_{(w)} {}^3e_{(w)}^u \partial_u {}^3e_{(b)r} + {}^3e_{(b)u} \partial_r (N_{(w)} {}^3e_{(w)}^u)) \Big) - \\
&\quad - {}^3e_{(b)}^s \left(\partial_r (N_{(w)} {}^3e_{(w)}^u \partial_u {}^3e_{(a)s} + {}^3e_{(a)u} \partial_s (N_{(w)} {}^3e_{(w)}^u)) - \right.
\end{aligned}$$

$$\begin{aligned}
& -\partial_s(N_{(w)}{}^3e_{(w)}^u\partial_u{}^3e_{(a)r}+{}^3e_{(a)u}\partial_r(N_{(w)}{}^3e_{(w)}^u))) + \\
& +{}^3e_{(a)}^u{}^3e_{(b)}^v{}^3e_{(c)r}(\partial_v(N_{(w)}{}^3e_{(w)}^t\partial_t{}^3e_{(c)u}+{}^3e_{(c)t}\partial_u(N_{(w)}{}^3e_{(w)}^t))) - \\
& -\partial_u(N_{(w)}{}^3e_{(w)}^t\partial_t{}^3e_{(c)v}+{}^3e_{(c)t}\partial_v(N_{(w)}{}^3e_{(w)}^t))) + \\
& +\left[(\partial_r{}^3e_{(b)s}-\partial_s{}^3e_{(b)r})\epsilon_{(a)(m)(n)}-(\partial_r{}^3e_{(a)s}-\partial_s{}^3e_{(a)r})\epsilon_{(b)(m)(n)}\right]{}^3e_{(n)}^s + \\
& +(\partial_v{}^3e_{(c)u}-\partial_u{}^3e_{(c)v})\left[{}^3e_{(b)}^v{}^3e_{(c)r}\epsilon_{(a)(m)(n)}{}^3e_{(n)}^u + \right. \\
& \left. +{}^3e_{(a)}^u{}^3e_{(c)r}\epsilon_{(b)(m)(n)}{}^3e_{(n)}^v +{}^3e_{(a)}^u{}^3e_{(b)}^v\epsilon_{(c)(m)(n)}{}^3e_{(n)r}\right]\hat{\mu}_{(m)} + \\
& +\left[{}^3e_{(a)}^s\epsilon_{(b)(m)(n)}-{}^3e_{(b)}^s\epsilon_{(a)(m)(n)}\right]\left[\partial_r(\hat{\mu}_{(m)}{}^3e_{(n)s})-\partial_s(\hat{\mu}_{(m)}{}^3e_{(n)r})\right] + \\
& +{}^3e_{(a)}^u{}^3e_{(b)}^v{}^3e_{(c)r}\epsilon_{(c)(m)(n)}\left[\partial_v(\hat{\mu}_{(m)}{}^3e_{(n)u})-\partial_u(\hat{\mu}_{(m)}{}^3e_{(n)v})\right]. \tag{3.21}
\end{aligned}$$

C. Comparison with Other Approaches to Tetrad Gravity.

Let us consider the canonical transformation

$$\begin{aligned}
& \tilde{\pi}^N(\tau, \vec{\sigma}) dN(\tau, \vec{\sigma}) + \tilde{\pi}_{(a)}^{\vec{N}}(\tau, \vec{\sigma}) dN_{(a)}(\tau, \vec{\sigma}) + \tilde{\pi}_{(a)}^{\vec{\varphi}}(\tau, \vec{\sigma}) d\varphi_{(a)}(\tau, \vec{\sigma}) + \\
& + {}^3\tilde{\pi}_{(a)}^r(\tau, \vec{\sigma}) d{}^3e_{(a)r}(\tau, \vec{\sigma}) = {}^4\tilde{\pi}_{(a)}^A(\tau, \vec{\sigma}) d{}^4E_A^{(\alpha)}(\tau, \vec{\sigma}), \tag{3.22}
\end{aligned}$$

where ${}^4\tilde{\pi}_{(a)}^A$ ⁴⁹ would be the canonical momenta if the ADM action would be considered as a functional of the cotetrads ${}^4E_A^{(\alpha)} = {}^4E_\mu^{(\alpha)} b_A^\mu$ in the holonomic Σ_τ -adapted basis, as essentially is done in Refs. [26,25]. If $\bar{\gamma} = \sqrt{1 + \sum_{(c)} \varphi^{(c)2}}$, we have

$$\begin{aligned}
\tilde{\pi}^N &= (\bar{\gamma} {}^4\tilde{\pi}_{(o)}^\tau + \varphi^{(a)} {}^4\tilde{\pi}_{(a)}^\tau), \\
\tilde{\pi}_{(a)}^{\vec{N}} &= -\epsilon\varphi_{(a)} {}^4\tilde{\pi}_{(o)}^\tau + [\delta_{(a)}^{(b)} - \epsilon\frac{\varphi_{(a)}\varphi^{(b)}}{1+\bar{\gamma}}] {}^4\tilde{\pi}_{(b)}^\tau, \\
\tilde{\pi}_{(a)}^{\vec{\varphi}} &= (\frac{\epsilon N}{\bar{\gamma}} \varphi_{(a)} - N_{(a)}) {}^4\tilde{\pi}_{(o)}^\tau - \delta_{(a)}^{(b)} N {}^4\tilde{\pi}_{(b)}^\tau - {}^3e_{(a)r} {}^4\tilde{\pi}_{(o)}^r - \\
& - \frac{1}{1+\bar{\gamma}} (\delta_{(a)}^{(c)} \varphi^{(b)} + \delta_{(a)}^{(b)} \varphi^{(c)} + \epsilon\frac{\varphi_{(a)}\varphi^{(b)}\varphi^{(c)}}{\bar{\gamma}(1+\bar{\gamma})}) (N_{(c)} {}^4\tilde{\pi}_{(b)}^\tau + {}^3e_{(c)r} {}^4\tilde{\pi}_{(b)}^r), \\
{}^3\tilde{\pi}_{(a)}^r &= -\epsilon\varphi_{(a)} {}^4\tilde{\pi}_{(o)}^r + (\delta_{(a)}^{(b)} - \epsilon\frac{\varphi_{(a)}\varphi^{(b)}}{1+\bar{\gamma}}) {}^4\tilde{\pi}_{(b)}^r, \\
{}^4\tilde{\pi}_{(o)}^\tau &= \bar{\gamma}\tilde{\pi}^N - \varphi^{(a)} {}^3\tilde{\pi}_{(a)}^{\vec{N}}, \\
{}^4\tilde{\pi}_{(a)}^\tau &= \epsilon\varphi_{(a)}\tilde{\pi}^N + [\delta_{(a)}^{(b)} - \epsilon\frac{\varphi_{(a)}\varphi^{(b)}}{1+\bar{\gamma}}]\tilde{\pi}_{(b)}^{\vec{N}}, \\
{}^4\tilde{\pi}_{(o)}^r &= -\bar{\gamma}{}^3e_{(a)}^r[\delta_{(a)}^{(b)} - \epsilon\frac{\varphi_{(a)}\varphi^{(b)}}{1+\bar{\gamma}}]\tilde{\pi}_{(b)}^{\vec{\varphi}} + \bar{\gamma}N_{(a)}{}^3e_{(a)}^r\tilde{\pi}^N +
\end{aligned}$$

⁴⁹ $\{ {}^4E_A^{(\alpha)}(\tau, \vec{\sigma}), {}^4\tilde{\pi}_{(\beta)}^B(\tau, \vec{\sigma}') \} = \delta_A^B \delta_{(\beta)}^{(\alpha)} \delta^3(\vec{\sigma}, \vec{\sigma}')$.

$$\begin{aligned}
& + {}^3e_{(a)}^r [-N \delta_{(a)}^{(b)} - (\delta_{(a)}^{(b)} \varphi^{(c)} - \delta_{(a)}^{(c)} \varphi^{(b)}) \frac{N_{(c)}}{1 + \bar{\gamma}}] \tilde{\pi}_{(b)}^{\vec{N}} - \\
& - \frac{1}{1 + \bar{\gamma}} {}^3e_{(a)}^r [\delta_{(a)}^{(b)} \varphi^{(c)} + \frac{1}{\bar{\gamma}} \delta_{(b)}^{(c)} \varphi^{(a)}] {}^3e_{(c)s} {}^3\tilde{\pi}_{(b)}^s, \\
{}^4\tilde{\pi}_{(a)}^r & = [\delta_{(a)}^{(b)} + \epsilon \frac{\varphi_{(a)} \varphi^{(b)}}{\bar{\gamma}(1 + \bar{\gamma})}] {}^3\tilde{\pi}_{(b)}^r - \epsilon \varphi_{(a)} {}^3e_{(b)}^r [\delta_{(b)}^{(c)} - \epsilon \frac{\varphi_{(b)} \varphi^{(c)}}{1 + \bar{\gamma}}] \tilde{\pi}_{(c)}^{\vec{\varphi}} + \\
& + \epsilon \varphi_{(a)} {}^3e_{(b)}^r N_{(b)} \tilde{\pi}^N - \epsilon \varphi_{(a)} {}^3e_{(b)}^r [N \delta_{(c)}^{(b)} + (\delta_{(c)}^{(b)} \varphi^{(d)} - \delta_{(c)}^{(d)} \varphi^{(b)}) \frac{N_{(d)}}{1 + \bar{\gamma}}] \tilde{\pi}_{(c)}^{\vec{N}} - \\
& - \epsilon \frac{\varphi_{(a)}}{1 + \bar{\gamma}} [\varphi^{(b)} \delta_{(d)}^{(c)} + \frac{1}{\bar{\gamma}} \varphi^{(c)} \delta_{(d)}^{(b)}] {}^3e_{(c)}^r {}^3e_{(b)s} {}^3\tilde{\pi}_{(d)}^s. \tag{3.23}
\end{aligned}$$

Our canonical transformation (3.23) allows to consider the metric ADM Lagrangian as function of the cotetrads ${}^4E_A^{(\alpha)} = {}^4E_\mu^{(\alpha)} b_A^\mu$ and to find the conjugate momenta ${}^4\tilde{\pi}_{(\alpha)}^A$. Eqs.(3.23) show that the four primary constraints, which contain the informations $\tilde{\pi}^N \approx 0$ and $\tilde{\pi}_{(a)}^{\vec{N}} \approx 0$, are ${}^4\tilde{\pi}_{(\alpha)}^r \approx 0$. The six primary constraints ${}^4\tilde{M}_{(\alpha)(\beta)} = {}^4E_A^{(\gamma)} [{}^4\eta_{(\alpha)(\gamma)} {}^4\tilde{\pi}_{(\beta)}^A - {}^4\eta_{(\beta)(\gamma)} {}^4\tilde{\pi}_{(\alpha)}^A] \approx 0$, generators of the local Lorentz transformations in this formulation, have the following relation with $\tilde{\pi}_{(a)}^{\vec{\varphi}} \approx 0$ and ${}^3\tilde{M}_{(a)} \approx 0$

$$\begin{aligned}
{}^4\tilde{M}_{(a)(b)} & = -\epsilon {}^3\tilde{M}_{(a)(b)} + (\varphi_{(a)} \tilde{\pi}_{(b)}^{\vec{\varphi}} - \varphi_{(b)} \tilde{\pi}_{(a)}^{\vec{\varphi}}) + \epsilon (\varphi_{(a)} N_{(b)} - \varphi_{(b)} N_{(a)}) \tilde{\pi}^N - \\
& - (\delta_{(a)}^{(c)} \delta_{(b)}^{(d)} - \delta_{(b)}^{(c)} \delta_{(a)}^{(d)}) [\epsilon N \varphi_{(c)} \delta_{(d)(e)} + \\
& + (\delta_{(c)(f)} + \frac{\varphi_{(c)} \varphi_{(f)}}{1 + \bar{\gamma}}) (\delta_{(d)(e)} + \frac{\varphi_{(d)} \varphi_{(e)}}{1 + \bar{\gamma}}) N_{(f)}] \tilde{\pi}_{(e)}^{\vec{N}} \approx 0, \\
{}^4\tilde{M}_{(a)(o)} & = -\epsilon \bar{\gamma} \tilde{\pi}_{(a)}^{\vec{\varphi}} - \frac{1}{1 + \bar{\gamma}} {}^3\tilde{M}_{(a)(b)} \varphi_{(b)} - \epsilon \bar{\gamma} (\delta_{(a)(b)} - \frac{\varphi_{(a)} \varphi_{(b)}}{\bar{\gamma}(1 + \bar{\gamma})}) N_{(b)} \tilde{\pi}^N + \\
& + [-\epsilon \bar{\gamma} N (\delta_{(a)(b)} - \frac{\varphi_{(a)} \varphi_{(b)}}{\bar{\gamma}(1 + \bar{\gamma})}) + \varphi_{(c)} N_{(c)} \delta_{(a)(b)} - N_{(a)} \varphi_{(b)}] \tilde{\pi}_{(b)}^{\vec{N}} \approx 0, \\
{}^3\tilde{M}_{(a)(b)} & = -\epsilon {}^4\tilde{M}_{(a)(b)} + \frac{\epsilon}{1 + \bar{\gamma}} [\varphi_{(a)} {}^4\tilde{M}_{(b)(c)} - \varphi_{(b)} {}^4\tilde{M}_{(a)(c)}] \varphi_{(c)} + \\
& + [\varphi_{(a)} {}^4\tilde{M}_{(b)(o)} - \varphi_{(b)} {}^4\tilde{M}_{(a)(o)}] - [\varphi_{(a)} {}^4E_{(b)}^\tau - \varphi_{(b)} {}^4E_{(a)}^\tau] {}^4\tilde{\pi}_{(o)}^\tau - \\
& - \epsilon [(\delta_{(a)(c)} \delta_{(d)(e)} - \delta_{(a)(e)} \delta_{(b)(c)}) (\delta_{(c)(d)} + \frac{\varphi_{(c)} \varphi_{(d)}}{1 + \bar{\gamma}}) {}^4E_{(c)}^\tau + \\
& + \epsilon ({}^4E_{(o)}^\tau + \epsilon \frac{\varphi_{(c)} {}^4E_{(c)}^\tau}{1 + \bar{\gamma}}) (\delta_{(a)(d)} \varphi_{(b)} - \delta_{(b)(d)} \varphi_{(a)})] {}^4\tilde{\pi}_{(d)}^\tau \approx 0, \\
\tilde{\pi}_{(a)}^{\vec{\varphi}} & = \epsilon (\delta_{(a)(b)} - \frac{\varphi_{(a)} \varphi_{(b)}}{\bar{\gamma}(1 + \bar{\gamma})}) {}^4\tilde{M}_{(b)(o)} + \frac{1}{1 + \bar{\gamma}} {}^4\tilde{M}_{(a)(b)} \varphi_{(b)} - \\
& - \epsilon (\delta_{(a)(b)} - \frac{\varphi_{(a)} \varphi_{(b)}}{\bar{\gamma}(1 + \bar{\gamma})}) {}^4E_{(b)}^\tau {}^4\tilde{\pi}_{(o)}^\tau + \\
& + [\epsilon (\delta_{(a)(b)} - \frac{\varphi_{(a)} \varphi_{(b)}}{\bar{\gamma}(1 + \bar{\gamma})}) {}^4E_{(o)}^\tau - \frac{\varphi_{(c)}}{1 + \bar{\gamma}} (\delta_{(c)(b)} {}^4E_{(a)}^\tau - \delta_{(c)(a)} {}^4E_{(b)}^\tau)] {}^4\tilde{\pi}_{(b)}^\tau \approx 0. \tag{3.24}
\end{aligned}$$

Let us add a comment on the literature on tetrad gravity. The use of tetrads started with Ref. [16], where vierbeins and spin connections are used as independent variables in a Palatini form of the Lagrangian. They were used by Dirac [17] for the coupling of gravity to

fermion fields (see also Ref. [24]) and here Σ_r -adapted tetrads were introduced. In Ref. [18] the reduction of this theory at the Lagrangian level was done by introducing the so-called *time-gauge* ${}^4E_r^{(o)} = 0$ [or ${}^4E_{(a)}^o = 0$], which distinguishes the time coordinate $x^o = \text{const.}$ planes; in this paper there is also the coupling to scalar fields, while in Ref. [19] the coupling to Dirac-Majorana fields is studied. In Ref. [21] there is a non-metric Lagrangian formulation, see Eq.(A33), employing as basic variables the cotetrads ${}^4E_\mu^{(\alpha)}$, which is different from our metric Lagrangian and has different primary constraints; its Hamiltonian formulation is completely developed. See also Ref. [22] for a study of the tetrad frame constraint algebra. In the fourth of Refs. [20] cotetrads ${}^4E_\mu^{(\alpha)}$ together with the spin connection ${}^4\omega_{\mu(\beta)}^{(\alpha)}$ are used as independent variables in a first order Palatini action ⁵⁰, while in Ref. [23] a first order Lagrangian reformulation is done for Eq.(A33) ⁵¹.

Instead in most of Refs. [20,25,26] one uses the space components ${}^4E_r^{(\alpha)}$ of cotetrads ${}^4E_\mu^{(\alpha)}$, together with the conjugate momenta ${}^4\tilde{\pi}_{(\alpha)}^r$ inside the ADM Hamiltonian, in which one puts ${}^3g_{rs} = {}^4E_r^{(\alpha)} {}^4\eta_{(\alpha)(\beta)} {}^4E_s^{(\beta)}$ and ${}^3\tilde{\Pi}^{rs} = \frac{1}{4} {}^4\eta^{(\alpha)(\beta)} [{}^4E_{(\alpha)}^r {}^4\tilde{\pi}_{(\beta)}^s + {}^4E_{(\alpha)}^s {}^4\tilde{\pi}_{(\beta)}^r]$. Lapse and shift functions are treated as Hamiltonian multipliers and there is no worked out Lagrangian formulation. In Ref. [27] it is shown how to go from the space components ${}^4E_r^{(\alpha)}$ to cotriads ${}^3e_{(a)r}$ by using the *time gauge* on a surface $x^0 = \text{const.}$; here it is introduced for the first time the concept of parameters of Lorentz boosts ⁵², which was our starting point to arrive at the identification of the Wigner boost parameters $\varphi_{(a)}$. Finally in Ref. [63] there is a 3+1 decomposition of tetrads and cotetrads in which some boost-like parameters have been fixed (it is a Schwinger time gauge) so that one can arrive at a Lagrangian (different from ours) depending only on lapse, shift and cotriads.

In Ref. [27] there is another canonical transformation from cotriads and their conjugate momenta to a new canonical basis containing densitized triads and their conjugate momenta

$$\begin{aligned} ({}^3e_{(a)r}, {}^3\tilde{\pi}_{(a)}^r) &\mapsto ({}^3\tilde{h}_{(a)}^r = {}^3e {}^3e_{(a)}^r, \\ {}^3K_{(a)r} &= 2[{}^3e_{(a)}^s {}^3K_{sr} + \frac{1}{4} {}^3\tilde{M}_{(a)(b)} {}^3e_{(b)r}] = \\ &= \frac{1}{2} [\frac{1}{k} {}^3G_{o(a)(b)(c)(d)} {}^3e_{(b)r} {}^3e_{(c)u} {}^3\tilde{\pi}_{(d)}^u + \frac{1}{3e} {}^3\tilde{M}_{(a)(b)} {}^3e_{(b)r}], \end{aligned} \quad (3.25)$$

which is used to make the transition to the complex Ashtekar variables [36]

$$({}^3\tilde{h}_{(a)}^r, {}^3A_{(a)r} = 2 {}^3K_{(a)r} + i {}^3\omega_{r(a)}), \quad (3.26)$$

where ${}^3A_{(a)r}$ is a zero density whose real part (in this notation) can be considered the gauge potential of the Sen connection and plays an important role in the simplification of

⁵⁰See also the Nelson-Regge papers in Refs. [20] for a different approach, the so-called *covariant canonical formalism*.

⁵¹In both these papers there is a 3+1 decomposition of the tetrads different from our and, like in Ref. [23], use is done of the Schwinger time gauge to get free of three boost-like parameters.

⁵²If they are put equal to zero, one recovers *Schwinger's time gauge*.

the functional form of the constraints present in this approach; the conjugate variable is a density 1 $SU(2)$ soldering form.

IV. COMPARISON WITH ADM CANONICAL METRIC GRAVITY.

In this Section we give a brief review of the Hamiltonian formulation of ADM metric gravity (see Refs. [64,65,14,66,67]) to express its constraints in terms of those of Section III.

The ADM Lagrangian $S_{ADM} = \int d\tau L_{ADM}(\tau) = \int d\tau d^3\sigma \mathcal{L}_{ADM}(\tau, \vec{\sigma})$ given in Eq.(3.1) is expressed in terms of the independent variables N , $N_r = {}^3g_{rs}N^s$, ${}^3g_{rs}$.

The Euler-Lagrange equations are

$$\begin{aligned}
L_N &= \frac{\partial \mathcal{L}_{ADM}}{\partial N} - \partial_\tau \frac{\partial \mathcal{L}_{ADM}}{\partial \partial_\tau N} - \partial_r \frac{\partial \mathcal{L}_{ADM}}{\partial \partial_r N} = \\
&= -\epsilon k \sqrt{\gamma} [{}^3R - {}^3K_{rs} {}^3K^{rs} + ({}^3K)^2] = -2\epsilon k {}^4\bar{G}_l \stackrel{\circ}{=} 0, \\
L_{\vec{N}}^r &= \frac{\partial \mathcal{L}_{ADM}}{\partial N_r} - \partial_\tau \frac{\partial \mathcal{L}_{ADM}}{\partial \partial_\tau N_r} - \partial_s \frac{\partial \mathcal{L}_{ADM}}{\partial \partial_s N_r} = \\
&= 2\epsilon k [\sqrt{\gamma} ({}^3K^{rs} - {}^3g^{rs} {}^3K)]_{|s} = 2k {}^4\bar{G}_l^r \stackrel{\circ}{=} 0, \\
L_g^{rs} &= -\epsilon k \left[\frac{\partial}{\partial \tau} [\sqrt{\gamma} ({}^3K^{rs} - {}^3g^{rs} {}^3K)] - N \sqrt{\gamma} ({}^3R^{rs} - \frac{1}{2} {}^3g^{rs} {}^3R) + \right. \\
&\quad \left. + 2N \sqrt{\gamma} ({}^3K^{ru} {}^3K_u^s - {}^3K {}^3K^{rs}) + \frac{1}{2} N \sqrt{\gamma} [({}^3K)^2 - {}^3K_{uv} {}^3K^{uv}] {}^3g^{rs} + \right. \\
&\quad \left. + \sqrt{\gamma} ({}^3g^{rs} N^{[u}_{|u} - N^{r|s]} \right] = -\epsilon k N \sqrt{\gamma} {}^4\bar{G}^{rs} \stackrel{\circ}{=} 0,
\end{aligned} \tag{4.1}$$

and correspond to the Einstein equations in the form ${}^4\bar{G}_l \stackrel{\circ}{=} 0$, ${}^4\bar{G}_{lr} \stackrel{\circ}{=} 0$, ${}^4\bar{G}_{rs} \stackrel{\circ}{=} 0$, respectively. As said after Eq.(A10) the four contracted Bianchi identities imply that only two of the equations $L_g^{rs} \stackrel{\circ}{=} 0$ are independent.

The canonical momenta (densities of weight -1) are

$$\begin{aligned}
\tilde{\Pi}^N(\tau, \vec{\sigma}) &= \frac{\delta S_{ADM}}{\delta \partial_\tau N(\tau, \vec{\sigma})} = 0, \\
\tilde{\Pi}_{\vec{N}}^r(\tau, \vec{\sigma}) &= \frac{\delta S_{ADM}}{\delta \partial_\tau N_r(\tau, \vec{\sigma})} = 0, \\
{}^3\tilde{\Pi}^{rs}(\tau, \vec{\sigma}) &= \frac{\delta S_{ADM}}{\delta \partial_\tau {}^3g_{rs}(\tau, \vec{\sigma})} = \epsilon k [\sqrt{\gamma} ({}^3K^{rs} - {}^3g^{rs} {}^3K)](\tau, \vec{\sigma}), \\
{}^3K_{rs} &= \frac{\epsilon}{k \sqrt{\gamma}} [{}^3\tilde{\Pi}_{rs} - \frac{1}{2} {}^3g_{rs} {}^3\tilde{\Pi}], \quad {}^3\tilde{\Pi} = {}^3g_{rs} {}^3\tilde{\Pi}^{rs} = -2\epsilon k \sqrt{\gamma} {}^3K,
\end{aligned} \tag{4.2}$$

and satisfy the Poisson brackets

$$\begin{aligned}
\{N(\tau, \vec{\sigma}), \tilde{\Pi}^N(\tau, \vec{\sigma}')\} &= \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\{N_r(\tau, \vec{\sigma}), \tilde{\Pi}_{\vec{N}}^s(\tau, \vec{\sigma}')\} &= \delta_r^s \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\{{}^3g_{rs}(\tau, \vec{\sigma}), {}^3\tilde{\Pi}^{uv}(\tau, \vec{\sigma}')\} &= \frac{1}{2} (\delta_r^u \delta_s^v + \delta_r^v \delta_s^u) \delta^3(\vec{\sigma}, \vec{\sigma}').
\end{aligned} \tag{4.3}$$

Let us introduce the Wheeler- DeWitt supermetric

$${}^3G_{rstw}(\tau, \vec{\sigma}) = [{}^3g_{rt} {}^3g_{sw} + {}^3g_{rw} {}^3g_{st} - {}^3g_{rs} {}^3g_{tw}](\tau, \vec{\sigma}), \tag{4.4}$$

whose inverse is defined by the equations

$$\begin{aligned} \frac{1}{2} {}^3G_{rstw} \frac{1}{2} {}^3G^{twuv} &= \frac{1}{2} (\delta_r^u \delta_s^v + \delta_r^v \delta_s^u), \\ {}^3G^{twuv}(\tau, \vec{\sigma}) &= [{}^3g^{tu} {}^3g^{wv} + {}^3g^{tv} {}^3g^{wu} - 2 {}^3g^{tw} {}^3g^{uv}](\tau, \vec{\sigma}). \end{aligned} \quad (4.5)$$

Then we get

$$\begin{aligned} {}^3\tilde{\Pi}^{rs}(\tau, \vec{\sigma}) &= \frac{1}{2} \epsilon k \sqrt{\gamma} {}^3G^{rsuv}(\tau, \vec{\sigma}) {}^3K_{uv}(\tau, \vec{\sigma}), \\ {}^3K_{rs}(\tau, \vec{\sigma}) &= \frac{\epsilon}{2k\sqrt{\gamma}} {}^3G_{rsuv}(\tau, \vec{\sigma}) {}^3\tilde{\Pi}^{uv}(\tau, \vec{\sigma}), \\ [{}^3K^{rs} {}^3K_{rs} - ({}^3K)^2](\tau, \vec{\sigma}) &= \\ &= k^{-2} [\gamma^{-1} ({}^3\tilde{\Pi}^{rs} {}^3\tilde{\Pi}_{rs} - \frac{1}{2} ({}^3\tilde{\Pi})^2)](\tau, \vec{\sigma}) = (2k)^{-1} [\gamma^{-1} {}^3G_{rsuv} {}^3\tilde{\Pi}^{rs} {}^3\tilde{\Pi}^{uv}](\tau, \vec{\sigma}), \\ \partial_\tau {}^3g_{rs}(\tau, \vec{\sigma}) &= [N_{r|s} + N_{s|r} - \frac{\epsilon N}{k\sqrt{\gamma}} {}^3G_{rsuv} {}^3\tilde{\Pi}^{uv}](\tau, \vec{\sigma}). \end{aligned} \quad (4.6)$$

Since ${}^3\tilde{\Pi}^{rs} \partial_\tau {}^3g_{rs} = {}^3\tilde{\Pi}^{rs} [N_{r|s} + N_{s|r} - \frac{\epsilon N}{k\sqrt{\gamma}} {}^3G_{rsuv} {}^3\tilde{\Pi}^{uv}] = -2N_r {}^3\tilde{\Pi}^{rs}|_s - \frac{\epsilon N}{k\sqrt{\gamma}} {}^3G_{rsuv} {}^3\tilde{\Pi}^{rs} {}^3\tilde{\Pi}^{uv} + (2N_r {}^3\tilde{\Pi}^{rs})|_s$, we obtain the canonical Hamiltonian ⁵³

$$\begin{aligned} H_{(c)ADM} &= \int_S d^3\sigma [\tilde{\Pi}^N \partial_\tau N + \tilde{\Pi}_N^r \partial_\tau N_r + {}^3\tilde{\Pi}^{rs} \partial_\tau {}^3g_{rs}](\tau, \vec{\sigma}) - L_{ADM} = \\ &= \int_S d^3\sigma [\epsilon N (k\sqrt{\gamma} {}^3R - \frac{1}{2k\sqrt{\gamma}} {}^3G_{rsuv} {}^3\tilde{\Pi}^{rs} {}^3\tilde{\Pi}^{uv}) - 2N_r {}^3\tilde{\Pi}^{rs}|_s](\tau, \vec{\sigma}) + \\ &+ 2 \int_{\partial S} d^2\Sigma_s [N_r {}^3\tilde{\Pi}^{rs}](\tau, \vec{\sigma}), \end{aligned} \quad (4.7)$$

In the following discussion we shall omit the surface term.

The Dirac Hamiltonian is [the $\lambda(\tau, \vec{\sigma})$'s are arbitrary Dirac multipliers]

$$H_{(D)ADM} = H_{(c)ADM} + \int d^3\sigma [\lambda_N \tilde{\Pi}^N + \lambda_r^{\tilde{N}} \tilde{\Pi}_N^r](\tau, \vec{\sigma}). \quad (4.8)$$

The τ -constancy of the primary constraints ⁵⁴ generates four secondary constraints (all 4 are densities of weight -1) which correspond to the Einstein equations ${}^4\tilde{G}_{ll}(\tau, \vec{\sigma}) \stackrel{\circ}{=} 0$, ${}^4\tilde{G}_{lr}(\tau, \vec{\sigma}) \stackrel{\circ}{=} 0$ [see after Eqs.(A10)]

$$\begin{aligned} \tilde{\mathcal{H}}(\tau, \vec{\sigma}) &= \epsilon [k\sqrt{\gamma} {}^3R - \frac{1}{2k\sqrt{\gamma}} {}^3G_{rsuv} {}^3\tilde{\Pi}^{rs} {}^3\tilde{\Pi}^{uv}](\tau, \vec{\sigma}) = \\ &= \epsilon [\sqrt{\gamma} {}^3R - \frac{1}{k\sqrt{\gamma}} ({}^3\tilde{\Pi}^{rs} {}^3\tilde{\Pi}_{rs} - \frac{1}{2} ({}^3\tilde{\Pi})^2)](\tau, \vec{\sigma}) = \\ &= \epsilon k \{ \sqrt{\gamma} [{}^3R - ({}^3K_{rs} {}^3K^{rs} - ({}^3K)^2)] \}(\tau, \vec{\sigma}) \approx 0, \\ {}^3\tilde{\mathcal{H}}^r(\tau, \vec{\sigma}) &= -2 {}^3\tilde{\Pi}^{rs}|_s(\tau, \vec{\sigma}) = -2 [\partial_s {}^3\tilde{\Pi}^{rs} + {}^3\Gamma_{su}^r {}^3\tilde{\Pi}^{su}](\tau, \vec{\sigma}) = \\ &= -2\epsilon k \{ \partial_s [\sqrt{\gamma} ({}^3K^{rs} - {}^3g^{rs} {}^3K)] + {}^3\Gamma_{su}^r \sqrt{\gamma} ({}^3K^{su} - {}^3g^{su} {}^3K) \}(\tau, \vec{\sigma}) \approx 0, \end{aligned} \quad (4.9)$$

⁵³Since $N_r {}^3\tilde{\Pi}^{rs}$ is a vector density of weight -1, we have ${}^3\nabla_s (N_r {}^3\tilde{\Pi}^{rs}) = \partial_s (N_r {}^3\tilde{\Pi}^{rs})$.

⁵⁴ $\partial_\tau \tilde{\Pi}^N(\tau, \vec{\sigma}) = \{ \tilde{\Pi}^N(\tau, \vec{\sigma}), H_{(D)ADM} \} \approx 0$, $\partial_\tau \tilde{\Pi}_N^r(\tau, \vec{\sigma}) = \{ \tilde{\Pi}_N^r(\tau, \vec{\sigma}), H_{(D)ADM} \} \approx 0$.

so that we have

$$H_{(c)ADM} = \int d^3\sigma [N \tilde{\mathcal{H}} + N_r {}^3\tilde{\mathcal{H}}^r](\tau, \vec{\sigma}) \approx 0, \quad (4.10)$$

with $\tilde{\mathcal{H}}(\tau, \vec{\sigma}) \approx 0$ called the *superhamiltonian* constraint and ${}^3\tilde{\mathcal{H}}^r(\tau, \vec{\sigma}) \approx 0$ called the *super-momentum* constraints. See Ref. [68] for their interpretation as the generators of the change of the canonical data ${}^3g_{rs}$, ${}^3\tilde{\Pi}^{rs}$, under the normal and tangent deformations of the spacelike hypersurface Σ_τ which generate $\Sigma_{\tau+d\tau}$ ⁵⁵.

In $\tilde{\mathcal{H}}(\tau, \vec{\sigma}) \approx 0$ we can say that the term $-\epsilon k \sqrt{\gamma} ({}^3K_{rs} {}^3K^{rs} - {}^3K^2)$ is the kinetic energy and $\epsilon k \sqrt{\gamma} {}^3R$ the potential energy: in any Ricci flat spacetime (i.e. one satisfying Einstein's empty-space equations) the extrinsic and intrinsic scalar curvatures of any spacelike hypersurface Σ_τ are both equal to zero (also the converse is true [70]).

All the constraints are first class, because the only non-identically zero Poisson brackets correspond to the so called universal Dirac algebra [1]:

$$\begin{aligned} \{ {}^3\tilde{\mathcal{H}}_r(\tau, \vec{\sigma}), {}^3\tilde{\mathcal{H}}_s(\tau, \vec{\sigma}') \} &= \\ &= {}^3\tilde{\mathcal{H}}_r(\tau, \vec{\sigma}') \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^s} + {}^3\tilde{\mathcal{H}}_s(\tau, \vec{\sigma}) \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^r}, \\ \{ \tilde{\mathcal{H}}(\tau, \vec{\sigma}), {}^3\tilde{\mathcal{H}}_r(\tau, \vec{\sigma}') \} &= \tilde{\mathcal{H}}(\tau, \vec{\sigma}) \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^r}, \\ \{ \tilde{\mathcal{H}}(\tau, \vec{\sigma}), \tilde{\mathcal{H}}(\tau, \vec{\sigma}') \} &= [{}^3g^{rs}(\tau, \vec{\sigma}) {}^3\tilde{\mathcal{H}}_s(\tau, \vec{\sigma}) + \\ &+ {}^3g^{rs}(\tau, \vec{\sigma}') {}^3\tilde{\mathcal{H}}_s(\tau, \vec{\sigma}')] \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}')}{\partial \sigma^r}, \end{aligned} \quad (4.11)$$

with ${}^3\tilde{\mathcal{H}}_r = {}^3g_{rs} {}^3\tilde{\mathcal{H}}^r$ as the combination of the supermomentum constraints satisfying the algebra of 3-diffeomorphisms. In Ref. [68] it is shown that Eqs.(4.11) are sufficient conditions for the embeddability of Σ_τ into M^4 . In the second paper in Ref. [5] it is shown that the last two lines of the Dirac algebra are the equivalent in phase space of the contracted Bianchi identities ${}^4G^{\mu\nu}{}_{;\nu} \equiv 0$.

The Hamilton-Dirac equations are

$$\begin{aligned} \partial_\tau N(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{ N(\tau, \vec{\sigma}), H_{(D)ADM} \} = \lambda_N(\tau, \vec{\sigma}), \\ \partial_\tau N_r(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{ N_r(\tau, \vec{\sigma}), H_{(D)ADM} \} = \lambda_r^{\tilde{N}}(\tau, \vec{\sigma}), \\ \partial_\tau {}^3g_{rs}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{ {}^3g_{rs}(\tau, \vec{\sigma}), H_{(D)ADM} \} = [N_{r|s} + N_{s|r} - \frac{2\epsilon N}{k\sqrt{\gamma}} ({}^3\tilde{\Pi}_{rs} - \frac{1}{2} {}^3g_{rs} {}^3\tilde{\Pi})](\tau, \vec{\sigma}) = \\ &= [N_{r|s} + N_{s|r} - 2N {}^3K_{rs}](\tau, \vec{\sigma}), \end{aligned}$$

⁵⁵One thinks to Σ_τ as determined by a cloud of observers, one per space point; the idea of bifurcation and reencounter of the observers is expressed by saying that the data on Σ_τ (where the bifurcation took place) are propagated to some final $\Sigma_{\tau+d\tau}$ (where the reencounter arises) along different intermediate paths, each path being a monoparametric family of surfaces that fills the sandwich in between the two surfaces; embeddability of Σ_τ in M^4 becomes the synonymous with path independence; see also Ref. [69] for the connection with the theorema egregium of Gauss.

$$\begin{aligned}
\partial_\tau {}^3\tilde{\Pi}^{rs}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} \{ {}^3\tilde{\Pi}^{rs}(\tau, \vec{\sigma}), H_{(D)ADM} \} = \epsilon [N k \sqrt{\gamma} ({}^3R^{rs} - \frac{1}{2} {}^3g^{rs} {}^3R)](\tau, \vec{\sigma}) - \\
&- 2\epsilon [\frac{N}{k\sqrt{\gamma}} (\frac{1}{2} {}^3\tilde{\Pi} {}^3\tilde{\Pi}^{rs} - {}^3\tilde{\Pi}^r{}_u {}^3\tilde{\Pi}^{us})](\tau, \vec{\sigma}) - \\
&- \frac{\epsilon N}{2} \frac{{}^3g^{rs}}{k\sqrt{\gamma}} (\frac{1}{2} {}^3\tilde{\Pi}^2 - {}^3\tilde{\Pi}_{uv} {}^3\tilde{\Pi}^{uv})](\tau, \vec{\sigma}) + \\
&+ \mathcal{L}_{\tilde{N}} {}^3\tilde{\Pi}^{rs}(\tau, \vec{\sigma}) + \epsilon [k\sqrt{\gamma} (N^{[r|s} - {}^3g^{rs} N^{u|u})](\tau, \vec{\sigma}), \\
&\Downarrow \\
\partial_\tau {}^3K_{rs}(\tau, \vec{\sigma}) &\stackrel{\circ}{=} (N[{}^3R_{rs} + {}^3K {}^3K_{rs} - 2 {}^3K_{ru} {}^3K^u{}_s] - \\
&- N_{|s|r} + N^u{}_{|s} {}^3K_{ur} + N^u{}_{|r} {}^3K_{us} + N^u {}^3K_{rs|u}) (\tau, \vec{\sigma}), \\
\partial_\tau \gamma(\tau, \vec{\sigma}) &\stackrel{\circ}{=} (2\gamma[-N {}^3K + N^u{}_{|u}]) (\tau, \vec{\sigma}), \\
\partial_\tau {}^3K(\tau, \vec{\sigma}) &\stackrel{\circ}{=} (N[{}^3g^{rs} {}^3R_{rs} + ({}^3K)^2] - N_{|u}{}^u + N^u {}^3K_{|u}) (\tau, \vec{\sigma}), \tag{4.12}
\end{aligned}$$

with $\mathcal{L}_{\tilde{N}} {}^3\tilde{\Pi}^{rs} = -\sqrt{\gamma} {}^3\nabla_u (\frac{N^u}{\sqrt{\gamma}} {}^3\tilde{\Pi}^{rs}) + {}^3\tilde{\Pi}^{ur} {}^3\nabla_u N^s + {}^3\tilde{\Pi}^{us} {}^3\nabla_u N^r$.

Use the following variation $\delta(\sqrt{\gamma} {}^3R)(\tau, \vec{\sigma}) = \int d^3\sigma_1 \{ (\sqrt{\gamma} {}^3R)(\tau, \vec{\sigma}), {}^3\tilde{\Pi}^{rs}(\tau, \vec{\sigma}_1) \} \delta^3 g_{rs}(\tau, \vec{\sigma}_1) = \int d^3\sigma_1 \delta^3 g_{rs}(\tau, \vec{\sigma}_1) \{ [-\sqrt{\gamma} ({}^3R^{rs} - \frac{1}{2} {}^3g^{rs} {}^3R)](\tau, \vec{\sigma}) \delta^3(\vec{\sigma}, \vec{\sigma}_1) + [\sqrt{\gamma} {}^3\Gamma_{lm}^n ({}^3g^{rl} {}^3g^{sm} - {}^3g^{rs} {}^3g^{lm})](\tau, \vec{\sigma}_1) \frac{\partial \delta^3(\vec{\sigma}, \vec{\sigma}_1)}{\partial \sigma^n} + [\sqrt{\gamma} ({}^3g^{rl} {}^3g^{sm} - {}^3g^{rs} {}^3g^{lm})](\tau, \vec{\sigma}_1) \frac{\partial^2 \delta^3(\vec{\sigma}, \vec{\sigma}_1)}{\partial \sigma^l \partial \sigma^m} \}$.

Let us remark that the canonical transformation [${}^4g_{AB}$ and ${}^4g^{AB}$ are given in Eqs.(A3)] $\tilde{\Pi}^N dN + \tilde{\Pi}^r_{\tilde{N}} dN_r + {}^3\tilde{\Pi}^{rs} d^3g_{rs} = {}^4\tilde{\Pi}^{AB} d^4g_{AB}$ defines the following momenta conjugated to ${}^4g_{AB}$

$$\begin{aligned}
{}^4\tilde{\Pi}^{\tau\tau} &= \frac{\epsilon}{2N} \tilde{\Pi}^N, \\
{}^4\tilde{\Pi}^{\tau r} &= \frac{\epsilon}{2} (\frac{N^r}{N} \tilde{\Pi}^N - \tilde{\Pi}^r_{\tilde{N}}), \\
{}^4\tilde{\Pi}^{rs} &= \epsilon (\frac{N^r N^s}{2N} \tilde{\Pi}^N - {}^3\tilde{\Pi}^{rs}), \\
\{ {}^4g_{AB}(\tau, \vec{\sigma}), {}^4\tilde{\Pi}^{CD}(\tau, \vec{\sigma}') \} &= \frac{1}{2} (\delta_A^C \delta_B^D + \delta_A^D \delta_B^C) \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\tilde{\Pi}^N &= \frac{2\epsilon}{\sqrt{\epsilon^4 g^{\tau\tau}}} {}^4\tilde{\Pi}^{\tau\tau}, \\
\tilde{\Pi}^r_{\tilde{N}} &= 2\epsilon \frac{{}^4g^{\tau r}}{{}^4g^{\tau\tau}} {}^4\tilde{\Pi}^{\tau\tau} - 2\epsilon {}^4\tilde{\Pi}^{\tau r}, \\
{}^3\tilde{\Pi}^{rs} &= \epsilon \frac{{}^4g^{\tau r} {}^4g^{\tau s}}{({}^4g^{\tau\tau})^2} {}^4\tilde{\Pi}^{\tau\tau} - \epsilon {}^4\tilde{\Pi}^{rs}, \tag{4.13}
\end{aligned}$$

which would emerge if the ADM action would be considered function of ${}^4g_{AB}$ instead of N , N_r and ${}^3g_{rs}$.

The standard ADM momenta ${}^3\tilde{\Pi}^{rs}$, defined in Eq. (4.2), may now be expressed in terms of the cotriads and their conjugate momenta of the canonical formulation of tetrad gravity given in Section III:

$$\begin{aligned}
{}^3\tilde{\Pi}^{rs} &= \epsilon k \sqrt{\gamma} ({}^3K^{rs} - {}^3g^{rs} {}^3K) = \frac{1}{4} [{}^3e_{(a)}^r {}^3\tilde{\pi}_{(a)}^s + {}^3e_{(a)}^s {}^3\tilde{\pi}_{(a)}^r], \\
\Rightarrow {}^3\tilde{\Pi} &= {}^3\tilde{\Pi}^{rs} {}^3g_{rs} = -2\epsilon k \sqrt{\gamma} {}^3K = \frac{1}{2} {}^3e_{(a)r} {}^3\tilde{\pi}_{(a)}^r, \\
\{ {}^3g_{rs}(\tau, \vec{\sigma}) &= {}^3e_{(a)r}(\tau, \vec{\sigma}) {}^3e_{(a)s}(\tau, \vec{\sigma}), {}^3\tilde{\Pi}^{uv}(\tau, \vec{\sigma}') \} = \frac{1}{2} (\delta_r^u \delta_s^v + \delta_s^u \delta_r^v) \delta^3(\vec{\sigma}, \vec{\sigma}'), \\
\{ {}^3\tilde{\Pi}^{rs}(\tau, \vec{\sigma}), {}^3\tilde{\Pi}^{uv}(\tau, \vec{\sigma}') \} &= \frac{1}{8} \delta^3(\vec{\sigma}, \vec{\sigma}') \times \\
[{}^3g^{ru} {}^3e_{(a)}^v {}^3e_{(b)}^s + {}^3g^{rv} {}^3e_{(a)}^u {}^3e_{(b)}^s &+ {}^3g^{su} {}^3e_{(a)}^v {}^3e_{(b)}^r + {}^3g^{sv} {}^3e_{(a)}^u {}^3e_{(b)}^r](\tau, \vec{\sigma}) \cdot \\
{}^3\tilde{M}_{(a)(b)}(\tau, \vec{\sigma}) &\approx 0.
\end{aligned} \tag{4.14}$$

The fact that in tetrad gravity the last Poisson brackets is only weakly zero has been noted in Ref. [25].

Let us now consider the expression of the ADM supermomentum constraints in tetrad gravity. Since ${}^3e_{(b)u} {}^3\tilde{\Pi}^{us} = \frac{1}{4} {}^3e_{(b)u} [{}^3e_{(a)}^u {}^3\tilde{\pi}_{(a)}^s + {}^3e_{(a)}^s {}^3\tilde{\pi}_{(a)}^u] = \frac{1}{4} [{}^3\tilde{\pi}_{(b)}^s + {}^3e_{(a)}^s {}^3e_{(b)u} {}^3\tilde{\pi}_{(a)}^u] = \frac{1}{4} [{}^3\tilde{\pi}_{(b)}^s + {}^3e_{(a)}^s ({}^3e_{(a)u} {}^3\tilde{\pi}_{(b)}^u + {}^3\tilde{M}_{(b)(a)})] = \frac{1}{4} [2 {}^3\tilde{\pi}_{(b)}^s - {}^3e_{(a)}^s {}^3\tilde{M}_{(a)(b)}]$, we have

$$\begin{aligned}
{}^3\tilde{\Pi}^{rs}|_s &= \partial_s {}^3\tilde{\Pi}^{rs} + {}^3\Gamma_{su}^r {}^3\tilde{\Pi}^{us} = \\
&= \partial_s {}^3\tilde{\Pi}^{rs} + [\epsilon_{(a)(b)(c)} {}^3e_{(a)}^r {}^3\omega_{s(c)} - \partial_s {}^3e_{(b)}^r] {}^3e_{(b)u} {}^3\tilde{\Pi}^{us} = \\
&= \frac{1}{4} (\partial_s [{}^3e_{(a)}^r {}^3\tilde{\pi}_{(a)}^s + {}^3e_{(a)}^s {}^3\tilde{\pi}_{(a)}^r] - \\
&- [\epsilon_{(a)(c)(b)} {}^3e_{(a)}^r {}^3\omega_{s(c)} + \partial_s {}^3e_{(b)}^r] \cdot [2 {}^3\tilde{\pi}_{(b)}^s - {}^3e_{(d)}^s {}^3\tilde{M}_{(d)(b)}]) = \\
&= \frac{1}{4} \{ {}^3e_{(a)}^r [\partial_s {}^3\tilde{\pi}_{(a)}^s - 2\epsilon_{(a)(b)(c)} {}^3\omega_{s(b)} {}^3\tilde{\pi}_{(c)}^s] - {}^3\tilde{\pi}_{(a)}^s \partial_s {}^3e_{(a)}^r + \\
&+ \partial_s ({}^3e_{(a)}^s {}^3\tilde{\pi}_{(a)}^r) + [\epsilon_{(a)(c)(b)} {}^3e_{(a)}^r {}^3\omega_{s(c)} + \partial_s {}^3e_{(b)}^r] {}^3e_{(d)}^s {}^3\tilde{M}_{(d)(b)} \} = \\
&= \frac{1}{4} \{ 2 {}^3e_{(a)}^r \hat{\mathcal{H}}_{(a)} + \partial_s [{}^3e_{(a)}^s {}^3\tilde{\pi}_{(a)}^r - {}^3e_{(a)}^r {}^3\tilde{\pi}_{(a)}^s] - \\
&- [\epsilon_{(a)(b)(c)} {}^3e_{(a)}^r {}^3\omega_{s(b)} + \partial_s {}^3e_{(c)}^r] {}^3e_{(d)}^s {}^3\tilde{M}_{(c)(d)} \}.
\end{aligned} \tag{4.15}$$

Since ${}^3\tilde{\pi}_{(a)}^r = \frac{1}{2} {}^3e_{(b)}^r [{}^3e_{(b)u} {}^3\tilde{\pi}_{(a)}^u + {}^3e_{(a)u} {}^3\tilde{\pi}_{(b)}^u] - \frac{1}{2} {}^3\tilde{M}_{(a)(b)} {}^3e_{(b)}^r$, we get $\partial_s [{}^3e_{(a)}^s {}^3\tilde{\pi}_{(a)}^r - {}^3e_{(a)}^r {}^3\tilde{\pi}_{(a)}^s] = \partial_s [\frac{1}{2} ({}^3e_{(a)}^s {}^3e_{(b)}^r - {}^3e_{(a)}^r {}^3e_{(b)}^s) ({}^3e_{(b)u} {}^3\tilde{\pi}_{(a)}^u + {}^3e_{(a)u} {}^3\tilde{\pi}_{(b)}^u) - {}^3e_{(a)}^s {}^3e_{(b)}^r {}^3\tilde{M}_{(a)(b)}] = -\partial_s [{}^3e_{(a)}^s {}^3e_{(b)}^r {}^3\tilde{M}_{(a)(b)}]$, the ADM metric supermomentum constraints (4.9) are satisfied in the following form

$$\begin{aligned}
{}^3\tilde{\mathcal{H}}^r &= -2 {}^3\tilde{\Pi}^{rs}|_s = \frac{1}{2} \{ -2 {}^3e_{(a)}^r \hat{\mathcal{H}}_{(a)} + \partial_s [{}^3e_{(a)}^s {}^3e_{(b)}^r {}^3\tilde{M}_{(a)(b)}] + \\
&+ [\partial_s {}^3e_{(c)}^r - \epsilon_{(c)(b)(a)} {}^3\omega_{s(b)} {}^3e_{(a)}^r] {}^3e_{(d)}^s {}^3\tilde{M}_{(c)(d)} \} = \\
&= \frac{1}{2} \{ 2 {}^3e_{(a)}^r {}^3e_{(a)}^s {}^3\tilde{\Theta}_s + [{}^3e_{(a)}^r {}^3\omega_{s(b)} - {}^3e_{(b)}^r {}^3\omega_{s(a)}] {}^3e_{(a)}^s {}^3\tilde{M}_{(b)} + \\
&+ \epsilon_{(a)(b)(c)} {}^3e_{(b)}^r \partial_s [{}^3e_{(a)}^s {}^3\tilde{M}_{(c)}] \} \approx 0.
\end{aligned} \tag{4.16}$$

V. CONCLUSIONS.

Motivated by the attempt to get a unified description and a canonical reduction of the four interactions in the framework of Dirac-Bergmann theory of constraint (the presymplectic approach), with this paper we have begun an investigation of general relativity along these lines. A complete analysis of the canonical reduction of this theory using constraint theory is still lacking, probably due to the fact that it does not respect the requirement of manifest general covariance. Instead, the presymplectic approach is the natural one to get an explicit control on the degrees of freedom of theories described by singular Lagrangians at the Hamiltonian level.

We have reviewed the kinematical framework for tetrad gravity, natural for the coupling to fermion fields, on globally hyperbolic, asymptotically flat at spatial infinity spacetimes whose 3+1 decomposition may be obtained with simultaneity spacelike hypersurfaces Σ_τ diffeomorphic to R^3 .

Then, we have given a new parametrization of arbitrary cotetrads in terms of lapse and shift functions, of cotriads on Σ_τ and of three boost parameters. Such parametrized cotetrads are put in the ADM action for metric gravity to obtain the new Lagrangian for tetrad gravity. In the Hamiltonian formulation, we obtain 14 first class constraints, ten primary and four secondary ones, whose algebra is studied.

A comparison with other formulations of tetrad gravity and with the Hamiltonian ADM metric gravity has been done.

In future papers based on Ref. [55], we shall study the Hamiltonian group of gauge transformations induced by the first class constraints. Then, the multitemporal equations associated with the constraints generating space rotations and space diffeomorphisms on the cotriads will be studied and solved. The Dirac observables with respect to thirteen of the fourteen constraints will be found in 3-orthogonal coordinates on Σ_τ and the associated Shanmugadhasan canonical transformation will be done. The only left constraint to be studied will be the superhamiltonian one. Some interpretational problems (Dirac observables versus general covariance) [71,13] will be faced, since they are deeply different from their counterpart in ordinary gauge theories like Yang-Mills one.

APPENDIX A: NOTATIONS.

In this Appendix we shall introduce the notations needed to define the ADM tetrads and triads used in this paper together with a review of kinematical notations.

1. Pseudo-Riemannian Geometry.

Let M^4 be a torsion-free, globally hyperbolic, asymptotically flat pseudo-Riemannian (or Lorentzian) 4-manifold, whose non-degenerate 4-metric tensor ${}^4g_{\mu\nu}(x)$ has Lorentzian signature $\epsilon(+, -, -, -)$ with $\epsilon = \pm 1$ according to particle physics and general relativity conventions respectively; the inverse 4-metric is ${}^4g^{\mu\nu}(x)$ with ${}^4g^{\mu\rho}(x){}^4g_{\rho\nu}(x) = \delta^\mu_\nu$. We shall denote with Greek letters μ, ν, \dots ($\mu = 0, 1, 2, 3$), the world indices and with Greek letters inside round brackets $(\alpha), (\beta), \dots$, flat Minkowski indices⁵⁶; analogously, a, b, \dots , and $(a), (b), \dots$, $[a=1,2,3]$, will denote world and flat 3-space indices. We shall follow the conventions of Refs. [65,52] for $\epsilon = -1$ and those of Ref. [72] for $\epsilon = +1$ ⁵⁷.

The coordinates of a chart of the atlas of M^4 will be denoted $\{x^\mu\}$. M^4 is assumed to be orientable; its volume element in any right-handed coordinate basis is $-\eta\sqrt{{}^4g}d^4x$ ⁵⁸. In the coordinate bases $e_\mu = \partial_\mu$ and dx^μ for vector fields (TM^4) and one-forms (or covectors; T^*M^4) respectively, the unique metric-compatible Levi-Civita affine connection has the symmetric Christoffel symbols ${}^4\Gamma^\mu_{\alpha\beta} = {}^4\Gamma^\mu_{\beta\alpha} = \frac{1}{2}{}^4g^{\mu\nu}(\partial_\alpha {}^4g_{\beta\nu} + \partial_\beta {}^4g_{\alpha\nu} - \partial_\nu {}^4g_{\alpha\beta})$ as connection coefficients (${}^4\Gamma^\mu_{\mu\nu} = \partial_\nu \sqrt{{}^4g}$) and the associated covariant derivative is denoted ${}^4\nabla_\mu$ (or with a semicolon “;”): ${}^4V^\mu{}_{;\nu} = {}^4\nabla_\nu {}^4V^\mu = \partial_\nu {}^4V^\mu + {}^4\Gamma^\mu_{\nu\alpha} {}^4V^\alpha$, with the metric compatibility condition being ${}^4\nabla_\rho {}^4g^{\mu\nu} = 0$.

The Christoffel symbols are not tensors. If, instead of the chart of M^4 with coordinates $\{x^\mu\}$, we choose another chart of M^4 , overlapping with the previous one, with coordinates $\{x'^\mu = x'^\mu(x)\}$ ($x'^\mu(x)$ are smooth functions), in the overlap of the two charts we have the following transformation properties under general smooth coordinate transformations or diffeomorphisms of M^4 ($Diff M^4$) of ${}^4g_{\alpha\beta}(x)$ and of ${}^4\Gamma^\mu_{\alpha\beta}(x)$ respectively

$$\begin{aligned} {}^4g'_{\alpha\beta}(x'(x)) &= \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} {}^4g_{\mu\nu}(x), \\ {}^4\Gamma'_{\alpha\beta}{}^\mu(x'(x)) &= \frac{\partial x'^\mu}{\partial x^\nu} \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} {}^4\Gamma^\nu_{\gamma\delta}(x) + \frac{\partial^2 x^\nu}{\partial x'^\alpha \partial x'^\beta} \frac{\partial x'^\mu}{\partial x^\nu}. \end{aligned} \quad (A1)$$

For a tensor density of weight W , ${}^4\mathcal{T}^{\mu\dots}_{\alpha\dots} = ({}^4g)^{-W/2} {}^4T^{\mu\dots}_{\alpha\dots}$, we have ${}^4\mathcal{T}^{\mu\dots}_{\alpha\dots;\rho} = ({}^4g)^{-W/2} [({}^4g)^{W/2} {}^4T^{\mu\dots}_{\alpha\dots}]_{;\rho} = ({}^4g)^{-W/2} {}^4T^{\mu\dots}_{\alpha\dots;\rho} = \partial_\rho {}^4\mathcal{T}^{\mu\dots}_{\alpha\dots} + {}^4\Gamma^\mu_{\rho\nu} {}^4\mathcal{T}^{\nu\dots}_{\alpha\dots} + \dots -$

⁵⁶With flat 4-metric tensor ${}^4\eta_{(\alpha)(\beta)} = \epsilon(+, -, -, -)$ in Cartesian coordinates.

⁵⁷I.e. the conventions of standard textbooks; see also Ref. [37] for many results (this book is consistent with Ref. [65], even if its index conventions are different).

⁵⁸ η is a sign connected with the choice of the orientation and ${}^4g = |\det {}^4g_{\mu\nu}|$; with $\eta = \epsilon$ we get the choice of Ref. [65] for $\epsilon = -1$ and of Ref. [72] for $\epsilon = +1$.

${}^4\Gamma_{\rho\alpha}^\beta {}^4\mathcal{T}^{\mu\dots}_{\beta\dots} - \dots + W {}^4\Gamma_{\sigma\rho}^\sigma {}^4\mathcal{T}^{\mu\dots}_{\alpha\dots}$ ⁵⁹.
The Riemann curvature tensor is ⁶⁰

$${}^4R^\alpha_{\mu\beta\nu} = {}^4\Gamma_{\beta\rho}^\alpha {}^4\Gamma_{\nu\mu}^\rho - {}^4\Gamma_{\nu\rho}^\alpha {}^4\Gamma_{\beta\mu}^\rho + \partial_\beta {}^4\Gamma_{\mu\nu}^\alpha - \partial_\nu {}^4\Gamma_{\beta\mu}^\alpha,$$

$${}^4R_{\alpha\mu\beta\nu} = {}^4g_{\alpha\gamma} {}^4R^\gamma_{\mu\beta\nu} = -{}^4R_{\alpha\mu\nu\beta} = -{}^4R_{\mu\alpha\beta\nu} = {}^4R_{\beta\nu\alpha\mu},$$

$${}^4R_{\mu\nu} = {}^4R_{\nu\mu} = {}^4R^\beta_{\mu\beta\nu},$$

$${}^4R = {}^4g^{\mu\nu} {}^4R_{\mu\nu} = {}^4R^{\mu\nu}_{\mu\nu},$$

$${}^4R^\alpha_{\mu\beta\nu} + {}^4R^\alpha_{\beta\nu\mu} + {}^4R^\alpha_{\nu\mu\beta} \equiv 0,$$

$$({}^4\nabla_\gamma {}^4R)^\alpha_{\mu\beta\nu} + ({}^4\nabla_\beta {}^4R)^\alpha_{\mu\nu\gamma} + ({}^4\nabla_\nu {}^4R)^\alpha_{\mu\gamma\beta} \equiv 0,$$

$$\Rightarrow ({}^4\nabla_\gamma {}^4R^{(ricci)})_{\mu\nu} + ({}^4\nabla_\alpha {}^4R)^\alpha_{\mu\nu\gamma} - ({}^4\nabla_\nu {}^4R^{(ricci)})_{\mu\gamma} \equiv 0,$$

$$\Rightarrow {}^4\nabla_\mu {}^4G^{\mu\nu} \equiv 0, \quad {}^4G_{\mu\nu} = {}^4R_{\mu\nu} - \frac{1}{2} {}^4g_{\mu\nu} {}^4R, \quad {}^4G = -{}^4R. \quad (\text{A2})$$

We have also shown the Ricci tensor, the curvature scalar and the first and second Bianchi identities for the curvature tensor with their implications ⁶¹. There are 20 independent components of the Riemann tensor in four dimensions due to its symmetry properties.

Let our globally hyperbolic spacetime M^4 be foliated with spacelike Cauchy hypersurfaces Σ_τ , obtained with the embeddings $i_\tau : \Sigma \rightarrow \Sigma_\tau \subset M^4$, $\vec{\sigma} \mapsto x^\mu = z^\mu(\tau, \vec{\sigma})$, of a 3-manifold Σ in M^4 ⁶².

Let $n^\mu(\sigma)$ and $l^\mu(\sigma) = N(\sigma)n^\mu(\sigma)$ be the contravariant timelike normal and unit normal [${}^4g_{\mu\nu}(z(\sigma))l^\mu(\sigma)l^\nu(\sigma) = \epsilon$] to Σ_τ at the point $z(\sigma) \in \Sigma_\tau$. The positive function $N(\sigma) > 0$ is the *lapse* function: $N(\sigma)d\tau$ measures the proper time interval at $z(\sigma) \in \Sigma_\tau$ between Σ_τ and $\Sigma_{\tau+d\tau}$. The *shift* functions $N^r(\sigma)$ are defined so that $N^r(\sigma)d\tau$ describes the horizontal shift on Σ_τ such that, if $z^\mu(\tau + d\tau, \vec{\sigma} + d\vec{\sigma}) \in \Sigma_{\tau+d\tau}$,

$${}^{59}\partial_\rho ({}^4g)^{-W/2} + W ({}^4g)^{-W/2} {}^4\Gamma_{\mu\rho}^\mu = 0.$$

⁶⁰This is the definition of Ref. [65] for $\epsilon = -1$; for $\epsilon = +1$ it coincides with minus the definition of Ref. [72].

⁶¹ ${}^4G_{\mu\nu}$ is the Einstein tensor and ${}^4\nabla_\mu {}^4G^{\mu\nu} \equiv 0$ are called contracted Bianchi identities.

⁶² $\tau : M^4 \rightarrow R$ is a global, timelike, future-oriented function labelling the leaves of the foliation; x^μ are local coordinates in a chart of M^4 ; $\vec{\sigma} = \{\sigma^r\}$, $r=1,2,3$, are local coordinates in a chart of Σ , which is diffeomorphic to R^3 ; we shall use the notation $\sigma^A = (\sigma^\tau = \tau; \vec{\sigma})$, $A = \{\tau, r\}$, and $z^\mu(\sigma) = z^\mu(\tau, \vec{\sigma})$.

then $z^\mu(\tau + d\tau, \vec{\sigma} + d\vec{\sigma}) \approx z^\mu(\tau, \vec{\sigma}) + N(\tau, \vec{\sigma})d\tau l^\mu(\tau, \vec{\sigma}) + [d\sigma^r + N^r(\tau, \vec{\sigma})d\tau] \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^r}$; therefore, we have $\frac{\partial z^\mu(\sigma)}{\partial \tau} = N(\sigma)l^\mu(\sigma) + N^r(\sigma) \frac{\partial z^\mu(\tau, \vec{\sigma})}{\partial \sigma^r}$ for the so called *evolution* vector. For the covariant unit normal to Σ_τ we have $l_\mu(\sigma) = {}^4g_{\mu\nu}(z(\sigma))l^\nu(\sigma) = N(\sigma)\partial_\mu\tau|_{x=z(\sigma)}$.

Instead of local coordinates x^μ for M^4 , we use local coordinates σ^A on $R \times \Sigma \approx M^4$ [$x^\mu = z^\mu(\sigma)$ with inverse $\sigma^A = \sigma^A(x)$], i.e. a Σ_τ -adapted *holonomic coordinate basis* for vector fields $\partial_A = \frac{\partial}{\partial \sigma^A} \in T(R \times \Sigma) \mapsto b_A^\mu(\sigma)\partial_\mu = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A}\partial_\mu \in TM^4$, and for differential one-forms $dx^\mu \in T^*M^4 \mapsto d\sigma^A = b_A^\mu(\sigma)dx^\mu = \frac{\partial \sigma^A(z)}{\partial x^\mu}dx^\mu \in T^*(R \times \Sigma)$. Let us note that in the flat Minkowski spacetime the transformation coefficients $b_\mu^A(\sigma)$ and $b_A^\mu(\sigma)$ become the flat orthonormal cotetrads $z_\mu^A(\sigma) = \frac{\partial \sigma^A(x)}{\partial x^\mu}|_{x=z(\sigma)}$ and tetrads $z_A^\mu(\sigma) = \frac{\partial z^\mu(\sigma)}{\partial \sigma^A}$ of Ref. [7].

The induced 4-metric and inverse 4-metric become in the new basis

$${}^4g(x) = {}^4g_{\mu\nu}(x)dx^\mu \otimes dx^\nu = {}^4g_{AB}(z(\sigma))d\sigma^A \otimes d\sigma^B,$$

$$\begin{aligned} {}^4g_{\mu\nu} &= b_\mu^A {}^4g_{AB} b_\nu^B = \\ &= \epsilon(N^2 - {}^3g_{rs}N^rN^s)\partial_\mu\tau\partial_\nu\tau - \epsilon {}^3g_{rs}N^s(\partial_\mu\tau\partial_\nu\sigma^r + \partial_\nu\tau\partial_\mu\sigma^r) - \epsilon {}^3g_{rs}\partial_\mu\sigma^r\partial_\nu\sigma^s = \\ &= \epsilon l_\mu l_\nu - \epsilon {}^3g_{rs}(\partial_\mu\sigma^r + N^r\partial_\mu\tau)(\partial_\nu\sigma^s + N^s\partial_\nu\tau), \\ \Rightarrow {}^4g_{AB} &= \{{}^4g_{\tau\tau} = \epsilon(N^2 - {}^3g_{rs}N^rN^s); {}^4g_{\tau r} = -\epsilon {}^3g_{rs}N^s; {}^4g_{rs} = -\epsilon {}^3g_{rs}\} = \\ &= \epsilon[l_A l_B - {}^3g_{rs}(\delta_A^r + N^r\delta_A^\tau)(\delta_B^s + N^s\delta_B^\tau)], \end{aligned}$$

$$\begin{aligned} {}^4g^{\mu\nu} &= b_A^\mu {}^4g^{AB} b_B^\nu = \\ &= \frac{\epsilon}{N^2}\partial_\tau z^\mu\partial_\tau z^\nu - \frac{\epsilon N^r}{N^2}(\partial_\tau z^\mu\partial_r z^\nu + \partial_\tau z^\nu\partial_r z^\mu) - \epsilon({}^3g^{rs} - \frac{N^rN^s}{N^2})\partial_r z^\mu\partial_s z^\nu = \\ &= \epsilon[l^\mu l^\nu - {}^3g^{rs}\partial_r z^\mu\partial_s z^\nu], \\ \Rightarrow {}^4g^{AB} &= \{{}^4g^{\tau\tau} = \frac{\epsilon}{N^2}; {}^4g^{\tau r} = -\frac{\epsilon N^r}{N^2}; {}^4g^{rs} = -\epsilon({}^3g^{rs} - \frac{N^rN^s}{N^2})\} = \\ &= \epsilon[l^A l^B - {}^3g^{rs}\delta_r^A\delta_s^B], \end{aligned}$$

$$\begin{aligned} l^A &= l^\mu b_\mu^A = N {}^4g^{A\tau} = \frac{\epsilon}{N}(1; -N^r), \\ l_A &= l_\mu b_A^\mu = N\partial_A\tau = N\delta_A^\tau = (N; \vec{0}). \end{aligned} \tag{A3}$$

Here, we introduced the 3-metric of Σ_τ : ${}^3g_{rs} = -\epsilon {}^4g_{rs}$ with signature $(+++)$. If ${}^4\gamma^{rs}$ is the inverse of the spatial part of the 4-metric (${}^4\gamma^{ru} {}^4g_{us} = \delta_s^r$), the inverse of the 3-metric is ${}^3g^{rs} = -\epsilon {}^4\gamma^{rs}$ (${}^3g^{ru} {}^3g_{us} = \delta_s^r$). ${}^3g_{rs}(\tau, \vec{\sigma})$ are the components of the *first fundamental form* of the Riemann 3-manifold $(\Sigma_\tau, {}^3g)$ and we have

$$\begin{aligned} ds^2 &= {}^4g_{\mu\nu}dx^\mu dx^\nu = \epsilon(N^2 - {}^3g_{rs}N^rN^s)(d\tau)^2 - 2\epsilon {}^3g_{rs}N^s d\tau d\sigma^r - \epsilon {}^3g_{rs}d\sigma^r d\sigma^s = \\ &= \epsilon[N^2(d\tau)^2 - {}^3g_{rs}(d\sigma^r + N^r d\tau)(d\sigma^s + N^s d\tau)], \end{aligned} \tag{A4}$$

for the line element in M^4 . We must have $\epsilon {}^4g_{oo} > 0$, $\epsilon {}^4g_{ij} < 0$, $\left| \begin{smallmatrix} {}^4g_{ii} & {}^4g_{ij} \\ {}^4g_{ji} & {}^4g_{jj} \end{smallmatrix} \right| > 0$, $\epsilon \det {}^4g_{ij} > 0$.

If we define $g = {}^4g = |\det({}^4g_{\mu\nu})|$ and $\gamma = {}^3g = |\det({}^3g_{rs})|$, we also have

$$\begin{aligned} N &= \sqrt{\frac{{}^4g}{{}^3g}} = \frac{1}{\sqrt{{}^4g^{\tau\tau}}} = \sqrt{\frac{g}{\gamma}} = \sqrt{{}^4g_{\tau\tau} - \epsilon {}^3g^{rs} {}^4g_{\tau r} {}^4g_{\tau s}}, \\ N^r &= -\epsilon {}^3g^{rs} {}^4g_{\tau s} = -\frac{{}^4g^{\tau r}}{{}^4g^{\tau\tau}}, \quad N_r = {}^3g_{rs} N^s = -\epsilon {}^4g_{rs} N^s = -\epsilon {}^4g_{\tau r}. \end{aligned} \quad (\text{A5})$$

Let us remark (see Ref. [9]) that in the study of space and time measurements the equation $ds^2 = 0$ (use of light signals for the synchronization of clocks) and the definition $d\bar{\tau} = \sqrt{\epsilon {}^4g_{oo}} dx^o$ of proper time⁶³ imply the use in M^4 of a 3-metric ${}^3\tilde{\gamma}_{rs} = {}^4g_{rs} - \frac{{}^4g_{or} {}^4g_{os}}{{}^4g_{oo}} = -\epsilon({}^3g_{rs} + \frac{N_r N_s}{\epsilon {}^4g_{oo}})$ with the covariant shift functions $N_r = {}^3g_{rs} N^s = -\epsilon {}^4g_{or}$, which are connected with the conventionality of simultaneity [54] and with the direction dependence of the velocity of light ($c(\vec{n}) = \sqrt{\epsilon {}^4g_{oo}}/(1 + N_r n^r)$ in direction \vec{n}).

In the standard (not Hamiltonian) description of the 3+1 decomposition we utilize a Σ_τ -adapted non-holonomic non-coordinate basis $[\bar{A} = (l; r)]$

$$\begin{aligned} \hat{b}_{\bar{A}}^\mu(\sigma) &= \{\hat{b}_l^\mu(\sigma) = \epsilon l^\mu(\sigma) = N^{-1}(\sigma)[b_\tau^\mu(\sigma) - N^r(\sigma)b_r^\mu(\sigma)]; \\ &\quad \hat{b}_r^\mu(\sigma) = b_r^\mu(\sigma)\}, \\ \hat{b}_{\bar{\mu}}^{\bar{A}}(\sigma) &= \{\hat{b}_\mu^l(\sigma) = l_\mu(\sigma) = N(\sigma)b_\mu^\tau(\sigma) = N(\sigma)\partial_\mu\tau(z(\sigma)); \\ &\quad \hat{b}_\mu^r(\sigma) = b_\mu^r(\sigma) + N^r(\sigma)b_\mu^\tau(\sigma)\}, \\ \hat{b}_{\bar{\mu}}^{\bar{A}}(\sigma)\hat{b}_{\bar{A}}^{\bar{\nu}}(\sigma) &= \delta_{\bar{\mu}}^{\bar{\nu}}, \quad \hat{b}_{\bar{\mu}}^{\bar{A}}(\sigma)\hat{b}_{\bar{B}}^{\bar{\mu}}(\sigma) = \delta_{\bar{B}}^{\bar{A}}, \\ {}^4\bar{g}_{\bar{A}\bar{B}}(z(\sigma)) &= \hat{b}_{\bar{A}}^\mu(\sigma){}^4g_{\mu\nu}(z(\sigma))\hat{b}_{\bar{B}}^\nu(\sigma) = \\ &= \{{}^4\bar{g}_{ll}(\sigma) = \epsilon; {}^4\bar{g}_{lr}(\sigma) = 0; {}^4\bar{g}_{rs}(\sigma) = {}^4g_{rs}(\sigma) = -\epsilon {}^3g_{rs}\}, \\ {}^4\bar{g}^{\bar{A}\bar{B}} &= \{{}^4\bar{g}^{ll} = \epsilon; {}^4\bar{g}^{lr} = 0; {}^4\bar{g}^{rs} = {}^4g^{rs} = -\epsilon {}^3g^{rs}\}, \\ X_{\bar{A}} &= \hat{b}_{\bar{A}}^\mu \partial_\mu = \{X_l = \frac{1}{N}(\partial_\tau - N^r \partial_r); \partial_r\}, \\ \theta^{\bar{A}} &= \hat{b}_{\bar{\mu}}^{\bar{A}} dx^\mu = \{\theta^l = N d\tau; \theta^r = d\sigma^r + N^r d\tau\}, \\ \Rightarrow l_\mu(\sigma)b_r^\mu(\sigma) &= 0, \quad l^\mu(\sigma)b_\mu^r(\sigma) = -N^r(\sigma)/N(\sigma), \\ l^{\bar{A}} &= l^\mu \hat{b}_{\bar{\mu}}^{\bar{A}} = (\epsilon; l^r + N^r l^r) = (\epsilon; \vec{0}), \\ l_{\bar{A}} &= l_\mu \hat{b}_{\bar{A}}^\mu = (1; l_r) = (1; \vec{0}). \end{aligned} \quad (\text{A6})$$

The non-holonomic basis in Σ_τ -adapted coordinates is $\hat{b}_{\bar{A}}^{\bar{A}} = \hat{b}_{\bar{\mu}}^{\bar{A}} b_{\bar{A}}^\mu = \{\hat{b}_A^l = l_A; \hat{b}_A^r = \delta_A^r + N^r \delta_A^\tau\}$, $\hat{b}_{\bar{A}}^A = \hat{b}_{\bar{\mu}}^A b_{\bar{A}}^\mu = \{\hat{b}_l^A = \epsilon l^A; \hat{b}_r^A = \delta_r^A\}$.

⁶³ $\sqrt{\epsilon {}^4g_{oo}}$ determines the ratio between the rates of a standard clock at rest and a coordinate clock at the same point.

See Refs. [73,65,14] for the 3+1 decomposition of 4-tensors on M^4 . The *horizontal projector* ${}^3h_\mu^\nu = \delta_\mu^\nu - \epsilon l_\mu l^\nu$ on Σ_τ defines the 3-tensor fields on Σ_τ starting from the 4-tensor fields on M^4 . We have ${}^3h_{\mu\nu} = {}^4g_{\mu\nu} - \epsilon l_\mu l_\nu = -\epsilon {}^3g_{rs}(b_\mu^r + N^r b_\mu^\tau)(b_\nu^s + N^s b_\nu^\tau) = -\epsilon {}^3g_{rs} \hat{b}_\mu^r \hat{b}_\nu^s$ and for a 4-vector ${}^4V^\mu = {}^4V^{\bar{A}} \hat{b}_{\bar{A}}^\mu = {}^4V^l l^\mu + {}^4V^r \hat{b}_r^\mu$ we have ${}^3V^\mu = {}^3V^r \hat{b}_r^\mu = {}^3h_\nu^\mu {}^4V^\nu$, ${}^3V^r = \hat{b}_r^\mu {}^3V^\mu$.

The 3-dimensional covariant derivative (denoted ${}^3\nabla$ or with the subscript “|”) of a 3-dimensional tensor ${}^3T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q}$ of rank (p,q) is the 3-dimensional tensor of rank (p,q+1) ${}^3\nabla_\rho {}^3T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q} = {}^3T^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_q|\rho} = {}^3h_{\alpha_1}^{\mu_1} \dots {}^3h_{\alpha_p}^{\mu_p} {}^3h_{\nu_1}^{\beta_1} \dots {}^3h_{\nu_q}^{\beta_q} {}^3h_\rho^\sigma {}^4\nabla_\sigma {}^3T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}$. For (1,0) and (0,1) tensors we have: ${}^3\nabla_\rho {}^3V^\mu = {}^3V^\mu_{|\rho} = {}^3V^r_{|s} \hat{b}_r^\mu \hat{b}_\rho^s$, ${}^3\nabla_s {}^3V^r = {}^3V^r_{|s} = \partial_s {}^3V^r + {}^3\Gamma_{su}^r {}^3V^u$ and ${}^3\nabla_\rho {}^3\omega_\mu = {}^3\omega_{\mu|\rho} = {}^3\omega_{r|s} \hat{b}_\mu^r \hat{b}_\rho^s$, ${}^3\nabla_s {}^3\omega_r = {}^3\omega_{r|s} = \partial_s {}^3\omega_r - {}^3\Gamma_{rs}^u {}^3\omega_u$ respectively.

The 3-dimensional Christoffel symbols are ${}^3\Gamma_{rs}^u = \hat{b}_\mu^u [{}^3\nabla_\rho \hat{b}_r^\mu] \hat{b}_s^\rho = \hat{b}_\mu^u \hat{b}_{r|\rho}^\mu \hat{b}_s^\rho = \frac{1}{2} {}^3g^{uv} (\partial_s {}^3g_{vr} + \partial_r {}^3g_{vs} - \partial_v {}^3g_{rs})$ and the metric compatibility ⁶⁴ is ${}^3\nabla_\rho {}^3g_{\mu\nu} = {}^3g_{\mu\nu|\rho} = 0$ ⁶⁵. It is then possible to define parallel transport on Σ_τ .

The 3-dimensional curvature Riemann tensor is

$$\begin{aligned} {}^3R^\mu_{\alpha\nu\beta} {}^3V^\alpha &= {}^3V^\alpha_{|\beta|\nu} - {}^3V^\alpha_{|\nu|\beta}, \\ \Rightarrow {}^3R^r_{suv} &= \partial_u {}^3\Gamma_{sv}^r - \partial_v {}^3\Gamma_{su}^r + {}^3\Gamma_{uw}^r {}^3\Gamma_{sv}^w - {}^3\Gamma_{vw}^r {}^3\Gamma_{su}^w. \end{aligned} \quad (\text{A7})$$

For 3-manifolds, the Riemann tensor has only 6 independent components since the Weyl tensor vanishes: this gives the relation ${}^3R_{\alpha\mu\beta\nu} = \frac{1}{2}({}^3R_{\mu\beta} {}^3g_{\alpha\nu} + {}^3R_{\alpha\nu} {}^3g_{\mu\beta} - {}^3R_{\alpha\beta} {}^3g_{\mu\nu} - {}^3R_{\mu\nu} {}^3g_{\alpha\beta}) - \frac{1}{6}({}^3g_{\alpha\beta} {}^3g_{\mu\nu} - {}^3g_{\alpha\nu} {}^3g_{\beta\mu}) {}^3R$, which expresses the Riemann tensor in terms of the Ricci tensor.

The components of the *second fundamental form* of $(\Sigma_\tau, {}^3g)$ is the extrinsic curvature

$${}^3K_{\mu\nu} = {}^3K_{\nu\mu} = -\frac{1}{2} \mathcal{L}_l {}^3g_{\mu\nu}. \quad (\text{A8})$$

We have ${}^4\nabla_\rho l^\mu = \epsilon {}^3a^\mu l_\rho - {}^3K_\rho^\mu$, with the acceleration ${}^3a^\mu = {}^3a^r \hat{b}_r^\mu$ of the observers travelling along the congruence of timelike curves with tangent vector l^μ given by ${}^3a_r = \partial_r \ln N$. On Σ_τ we have

$${}^3K_{rs} = {}^3K_{sr} = \frac{1}{2N} (N_{r|s} + N_{s|r} - \frac{\partial {}^3g_{rs}}{\partial \tau}). \quad (\text{A9})$$

The information contained in the 20 independent components ${}^4R^\mu_{\nu\alpha\beta}$ of the curvature Riemann tensor of M^4 is given in terms of 3-tensors on Σ_τ by the following three projections ⁶⁶

$$\begin{aligned} {}^3h_\rho^\mu {}^3h_\nu^\sigma {}^3h_\alpha^\gamma {}^3h_\beta^\delta {}^4R^\rho_{\sigma\gamma\delta} &= {}^4\bar{R}^r_{suv} \hat{b}_r^\mu \hat{b}_\nu^s \hat{b}_\alpha^u \hat{b}_\beta^v = {}^3R^\mu_{\nu\alpha\beta} + {}^3K_\alpha^\mu {}^3K_{\beta\nu} - {}^3K_\beta^\mu {}^3K_{\alpha\nu}, \\ &\text{GAUSS EQUATION,} \end{aligned}$$

⁶⁴Levi-Civita connection on the Riemann 3-manifold $(\Sigma_\tau, {}^3g)$.

⁶⁵ ${}^3g_{\mu\nu} = -\epsilon {}^3h_{\mu\nu} = {}^3g_{rs} \hat{b}_\mu^r \hat{b}_\nu^s$, so that ${}^3\bar{g}_{\bar{A}\bar{B}} = \{{}^3\bar{g}_{ll} = 0; {}^3\bar{g}_{lr} = 0; {}^3\bar{g}_{rs} = -\epsilon {}^3g_{rs}\}$.

⁶⁶See Ref. [74] for the geometry of embeddings; one has ${}^4\bar{R}^r_{suv} = {}^3\bar{R}^r_{suv}$.

$$\begin{aligned}
& \epsilon l_\rho {}^3 h_\nu^\sigma {}^3 h_\alpha^\gamma {}^3 h_\beta^\delta {}^4 R^\rho{}_{\sigma\gamma\delta} = {}^4 \bar{R}^l{}_{su\nu} \hat{b}_\nu^s \hat{b}_\alpha^u \hat{b}_\beta^v = {}^3 K_{\alpha\nu|\beta} - {}^3 K_{\beta\nu|\alpha}, \\
& \text{CODAZZI - MAINARDI EQUATION,} \\
& {}^4 R_{\mu\sigma\gamma\delta} l^\sigma l^\gamma {}^3 h_\nu^\delta = {}^4 \bar{R}_{\mu ll u} \hat{b}_\nu^u = \epsilon ({}^3 \mathcal{L}_l {}^3 K_{\mu\nu} + {}^3 K_\mu{}^\rho {}^3 K_{\rho\nu} + {}^3 a_{\mu|\nu} + {}^3 a_\mu {}^3 a_\nu), \\
& \text{RICCI EQUATION,} \\
& \text{with} \quad \mathcal{L}_l {}^3 K_{\mu\nu} = l^\alpha {}^3 K_{\mu\nu;\alpha} - 2 {}^3 K_\mu{}^\alpha {}^3 K_{\alpha\nu} + 2\epsilon {}^3 a^\alpha {}^3 K_{\alpha(\nu} l_{\mu)}. \tag{A10}
\end{aligned}$$

After having expressed the 4-Riemann tensor components in the non-holonomic basis in terms of the 3-Riemann tensor on Σ_τ , the extrinsic curvature of Σ_τ and the acceleration⁶⁷, we can express ${}^4 R_{\mu\nu} = \epsilon {}^4 \bar{R}_{ll} l_\mu l_\nu + \epsilon {}^4 \bar{R}_{lr} (l_\mu \hat{b}_\nu^r + l_\nu \hat{b}_\mu^r) + {}^4 \bar{R}_{rs} \hat{b}_\mu^r \hat{b}_\nu^s$, ${}^4 R$ and the Einstein tensor ${}^4 G_{\mu\nu} = {}^4 R_{\mu\nu} - \frac{1}{2} {}^4 g_{\mu\nu} {}^4 R = \epsilon {}^4 \bar{G}_{ll} l_\mu l_\nu + \epsilon {}^4 \bar{G}_{lr} (l_\mu \hat{b}_\nu^r + l_\nu \hat{b}_\mu^r) + {}^4 \bar{G}_{rs} \hat{b}_\mu^r \hat{b}_\nu^s$. The vanishing of ${}^4 \bar{G}_{ll}$, ${}^4 \bar{G}_{lr}$, corresponds to the four secondary constraints (restrictions of Cauchy data) of the ADM Hamiltonian formalism (see Section IV). The four contracted Bianchi identities, ${}^4 G^{\mu\nu}{}_{;\nu} \equiv 0$, imply [37] that, if the restrictions of Cauchy data are satisfied initially and the spatial equations ${}^4 G_{ij} \stackrel{\circ}{=} 0$ are satisfied everywhere, then the secondary constraints are satisfied also at later times⁶⁸. The four contracted Bianchi identities plus the four secondary constraints imply that only two combinations of the Einstein equations ${}^4 \bar{G}_{rs} \stackrel{\circ}{=} 0$ are independent, namely contain the accelerations (second time derivatives) of the two (non tensorial) independent degrees of freedom of the gravitational field, and that only these two equations can be put in normal form⁶⁹.

The *intrinsic geometry* of Σ_τ is defined by the *Riemannian metric* ${}^3 g_{rs}$ ⁷⁰, the *Levi-Civita affine connection*, i.e. the Christoffel symbols ${}^3 \Gamma_{rs}^u$,⁷¹ and the *curvature Riemann tensor* ${}^3 R^r{}_{stu}$ ⁷². The *extrinsic geometry* of Σ_τ is defined by the *lapse* N and *shift* N^r fields, which describe the *evolution* of Σ_τ in M^4 , and by the *extrinsic curvature* ${}^3 K_{rs}$ ⁷³.

⁶⁷For instance ${}^4 R = {}^3 R + {}^3 K_{rs} {}^3 K^{rs} - ({}^3 K)^2$.

⁶⁸See Ref. [41,37] for the initial value problem.

⁶⁹This was one of the motivations behind the discovery of the Shanmugadhasan canonical transformations [4].

⁷⁰It allows to evaluate the length of space curves.

⁷¹For the parallel transport of 3-dimensional tensors on Σ_τ .

⁷²For the evaluation of the holonomy and for the geodesic deviation equation.

⁷³It is needed to evaluate how much a 3-dimensional vector goes outside Σ_τ under spacetime parallel transport and to rebuild the spacetime curvature from the 3-dimensional one.

2. Tetrads and Cotetrads on M^4 .

Besides the local dual coordinate bases ${}^4e_\mu = \partial_\mu$ and dx^μ for TM^4 and T^*M^4 respectively, we can introduce special *non-coordinate* bases ${}^4\hat{E}_{(\alpha)} = {}^4\hat{E}_{(\alpha)}^\mu(x)\partial_\mu$ and its dual ${}^4\hat{\theta}^{(\alpha)} = {}^4\hat{E}_{(\alpha)}^\mu(x)dx^\mu$ ⁷⁴ with the *vierbeins or tetrads or (local) frames* ${}^4\hat{E}_{(\alpha)}^\mu(x)$, which are, for each point $x^\mu \in M^4$, the matrix elements of matrices $\{{}^4\hat{E}_{(\alpha)}^\mu\} \in GL(4, R)$; the set of one-forms ${}^4\hat{\theta}^{(\alpha)}$ ⁷⁵ is also called *canonical* or *soldering* one-form or *coframe*. Since a *frame* ${}^4\hat{E}$ at the point $x^\mu \in M^4$ is a linear isomorphism [30] ${}^4\hat{E} : R^4 \rightarrow T_x M^4$, $\partial_\alpha \mapsto {}^4\hat{E}(\partial_\alpha) = {}^4\hat{E}_{(\alpha)}$, a frame determines a basis ${}^4\hat{E}_{(\alpha)}$ of $T_x M^4$ ⁷⁶ and we can define a principal fiber bundle with structure group $GL(4, R)$, $\pi : L(M^4) \rightarrow M^4$ called the *frame bundle* of M^4 ⁷⁷; if $\Lambda \in GL(4, R)$, then the free right action of $GL(4, R)$ on $L(M^4)$ is denoted $R_\Lambda({}^4\hat{E}) = {}^4\hat{E} \circ \Lambda$, ${}^4\hat{E}_{(\alpha)} \mapsto {}^4\hat{E}_{(\beta)} (\Lambda^{-1})^{(\beta)}_{(\alpha)}$. When M^4 is *parallelizable*⁷⁸, as we shall assume, then $L(M^4) = M^4 \times GL(4, R)$ is a trivial principal bundle⁷⁹. See Ref. [30] for the differential structure on $L(M^4)$.

With the assumed pseudo-Riemannian manifold $(M^4, {}^4g)$, we can use its metric ${}^4g_{\mu\nu}$ to define the *orthonormal frame bundle* of M^4 , $F(M^4) = M^4 \times SO(3, 1)$, with structure group $SO(3, 1)$, of the orthonormal frames (or *non-coordinate basis* or *orthonormal tetrads*) ${}^4E_{(\alpha)} = {}^4E_{(\alpha)}^\mu \partial_\mu$ of TM^4 . The orthonormal tetrads and their duals, the orthonormal cotetrads ${}^4E_{(\alpha)}^\mu$ ⁸⁰, satisfy the duality and orthonormality conditions

$$\begin{aligned} {}^4E_{(\alpha)}^\mu {}^4E_{(\beta)\mu} &= \delta_{(\alpha)}^{(\beta)}, & {}^4E_{(\alpha)}^\mu {}^4E_{(\alpha)\mu} &= \delta_\mu^\mu, \\ {}^4E_{(\alpha)}^\mu {}^4g_{\mu\nu} {}^4E_{(\beta)}^\nu &= {}^4\eta_{(\alpha)(\beta)}, & {}^4E_{(\alpha)}^\mu {}^4g^{\mu\nu} {}^4E_{(\beta)\nu} &= {}^4\eta^{(\alpha)(\beta)}. \end{aligned} \quad (A11)$$

Under a rotation $\Lambda \in SO(3, 1)$ ($\Lambda^4 \eta \Lambda^T = {}^4\eta$) we have ${}^4E_{(\alpha)}^\mu \mapsto {}^4E_{(\beta)}^\mu (\Lambda^{-1})^{(\beta)}_{(\alpha)}$, ${}^4E_{(\alpha)}^\mu \mapsto \Lambda^{(\alpha)}_{(\beta)} {}^4E_{(\beta)}^\mu$. Therefore, while the indices $\alpha, \beta \dots$ transform under general coordinate transformations (the diffeomorphisms in $Diff M^4$), the indices $(\alpha), (\beta) \dots$ transform

⁷⁴ $i_4 {}^4\hat{E}_{(\alpha)} {}^4\hat{\theta}^{(\beta)} = {}^4E_{(\alpha)}^\mu {}^4E_{(\beta)\mu} = \delta_{(\alpha)}^{(\beta)} \Rightarrow {}^4\eta_{(\alpha)(\beta)} = {}^4E_{(\alpha)}^\mu {}^4g_{\mu\nu} {}^4E_{(\beta)}^\nu$; $(\alpha) = (0), (1), (2), (3)$ are numerical indices.

⁷⁵With ${}^4\hat{E}_{(\alpha)}^\mu(x)$ being the dual *cotetrads*.

⁷⁶The coframes ${}^4\hat{\theta}$ determine a basis ${}^4\hat{\theta}^{(\alpha)}$ of $T_x^* M^4$.

⁷⁷Its fibers are the sets of all the frames over the points $x^\mu \in M^4$; it is an affine bundle, i.e. there is no (global when it exists) cross section playing the role of the identity cross section of vector bundles.

⁷⁸I.e. M^4 admits four vector fields which are independent in each point, so that the tangent bundle $T(M^4)$ is trivial, $T(M^4) = M^4 \times R^4$; this is not possible (no hair theorem) for any compact manifold except a torus.

⁷⁹I.e. it admits a global cross section $\sigma : M^4 \rightarrow L(M^4)$, $x^\mu \mapsto {}^4\hat{E}_{(\alpha)}(x)$.

⁸⁰ ${}^4\theta^{(\alpha)} = {}^4E_{(\alpha)}^\mu dx^\mu$ are the orthonormal coframes.

under Lorentz rotations. The 4-metric can be expressed in terms of orthonormal cotetrads or local coframes in the non-coordinate basis

$$\begin{aligned} {}^4g_{\mu\nu} &= {}^4E_{\mu}^{(\alpha)} {}^4\eta_{(\alpha)(\beta)} {}^4E_{\nu}^{(\beta)}, & {}^4g^{\mu\nu} &= {}^4E_{(\alpha)}^{\mu} {}^4\eta^{(\alpha)(\beta)} {}^4E_{(\beta)}^{\nu}, \\ {}^4g &= {}^4g_{\mu\nu} dx^{\mu} \otimes dx^{\nu} = {}^4\eta_{(\alpha)(\beta)} \theta^{(\alpha)} \otimes \theta^{(\beta)}. \end{aligned} \quad (\text{A12})$$

For each vector ${}^4V^{\mu}$ and covector ${}^4\omega_{\mu}$ we have the decompositions ${}^4V^{\mu} = {}^4V^{(\alpha)} {}^4E_{(\alpha)}^{\mu}$ (${}^4V^{(\alpha)} = {}^4E_{(\alpha)}^{\mu} {}^4V_{\mu}$), ${}^4\omega_{\mu} = {}^4E_{(\alpha)}^{\mu} {}^4\omega_{(\alpha)}$ (${}^4\omega_{(\alpha)} = {}^4E_{(\alpha)}^{\mu} {}^4\omega_{\mu}$).

In a non-coordinate basis we have

$$\begin{aligned} [{}^4E_{(\alpha)}, {}^4E_{(\beta)}] &= c_{(\alpha)(\beta)}^{(\gamma)} {}^4E_{(\gamma)}, \\ c_{(\alpha)(\beta)}^{(\gamma)} &= {}^4E_{\nu}^{(\gamma)} ({}^4E_{(\alpha)}^{\mu} \partial_{\mu} {}^4E_{(\beta)}^{\nu} - {}^4E_{(\beta)}^{\mu} \partial_{\mu} {}^4E_{(\alpha)}^{\nu}). \end{aligned} \quad (\text{A13})$$

Physically, in a coordinate system (chart) x^{μ} of M^4 , a tetrad may be considered as a collection of accelerated observers described by a congruence of timelike curves with 4-velocity ${}^4E_{(o)}^{\mu}$; in each point $p \in M^4$ consider a coordinate transformation to local inertial coordinates at p, i.e. $x^{\mu} \mapsto X_p^{(\mu)}(x)$: then we have, in p, ${}^4E_{(\alpha)}^{\mu}(p) = \frac{\partial x^{\mu}(X_p(p))}{\partial X_p^{(\alpha)}}$ and ${}^4E_{\mu}^{(\alpha)}(p) = \frac{\partial X_p^{(\alpha)}(p)}{\partial x^{\mu}}$ and locally we have a freely falling observer.

All the connection one-forms ω are 1-forms on the orthonormal frame bundle $F(M^4) = M^4 \times SO(3, 1)$. Since in general relativity we consider only Levi-Civita connections associated with pseudo-Riemannian 4-manifolds $(M^4, {}^4g)$, in $F(M^4)$ we consider only ω_{Γ} -horizontal subspaces H_{Γ} ⁸¹. Given a global cross section $\sigma : M^4 \rightarrow F(M^4) = M^4 \times SO(3, 1)$, the associated gauge potentials on M^4 , ${}^4\omega = \sigma^*\omega$, are the connection coefficients ${}^4\omega^{(T)} = \sigma^*\omega$ in the non-coordinate basis ${}^4E_{(\alpha)}$ ⁸²

$$\begin{aligned} {}^4\omega_{(\alpha)(\beta)}^{(T)(\gamma)} &= {}^4E_{\nu}^{(\gamma)} {}^4E_{(\alpha)}^{\mu} (\partial_{\mu} {}^4E_{(\beta)}^{\nu} + {}^4E_{(\beta)}^{\lambda} {}^4\Gamma_{\mu\lambda}^{(T)\nu}) = {}^4E_{\nu}^{(\gamma)} {}^4E_{(\alpha)}^{\mu} {}^4\nabla_{\mu} {}^4E_{(\beta)}^{\nu}, \\ {}^4\tilde{\nabla}_{{}^4E_{(\alpha)}} {}^4E_{(\beta)} &= {}^4\nabla_{{}^4E_{(\alpha)}} {}^4E_{(\beta)} - {}^4\omega_{(\alpha)(\beta)}^{(T)(\gamma)} {}^4E_{(\gamma)} = 0. \end{aligned} \quad (\text{A14})$$

The components of the Riemann tensors in the non-coordinate bases are ${}^4R^{(\alpha)}_{(\beta)(\gamma)(\delta)} = {}^4E_{(\gamma)}({}^4\omega_{(\delta)(\beta)}^{(T)(\alpha)}) - {}^4E_{(\delta)}({}^4\omega_{(\gamma)(\beta)}^{(T)(\alpha)}) + {}^4\omega_{(\delta)(\beta)}^{(T)(\epsilon)} {}^4\omega_{(\gamma)(\epsilon)}^{(T)(\alpha)} - {}^4\omega_{(\gamma)(\beta)}^{(T)(\epsilon)} {}^4\omega_{(\delta)(\epsilon)}^{(T)(\alpha)} - c_{(\gamma)(\delta)}^{(\epsilon)} {}^4\omega_{(\epsilon)(\beta)}^{(T)(\alpha)}$. The connection (gauge potential) one-form is ${}^4\omega^{(T)(\alpha)}_{(\beta)} = {}^4\omega_{(\gamma)(\beta)}^{(T)(\alpha)} {}^4\theta^{(\gamma)}$ ⁸³ and the curvature (field strength) 2-form is ${}^4\Omega^{(T)(\alpha)}_{(\beta)} = \frac{1}{2} {}^4\Omega^{(T)(\alpha)}_{(\beta)(\gamma)(\delta)} {}^4\theta^{(\gamma)} \wedge {}^4\theta^{(\delta)}$.

With the Levi-Civita connection⁸⁴, in a non-coordinate basis the spin connection takes the form

⁸¹ $TF(M^4) = V_{\Gamma} + H_{\Gamma}$ as a direct sum, with V_{Γ} the vertical subspace isomorphic to the Lie algebra $\mathfrak{o}(3,1)$ of $SO(3,1)$.

⁸²The second line defines them through the covariant derivative in the non-coordinate basis.

⁸³It is called improperly *spin connection*, while its components are called *Ricci rotation coefficients*.

⁸⁴It has zero torsion 2-form ${}^4T^{(\alpha)} = \frac{1}{2} T^{(\alpha)}_{(\beta)(\gamma)} {}^4\theta^{(\beta)} \wedge {}^4\theta^{(\gamma)} = 0$, namely ${}^4T^{(\alpha)}_{(\beta)(\gamma)} = {}^4\omega_{(\beta)(\gamma)}^{(T)(\alpha)} - {}^4\omega_{(\gamma)(\beta)}^{(T)(\alpha)} - c_{(\beta)(\gamma)}^{(\alpha)} = 0$.

$$\begin{aligned}
{}^4\omega^{(\alpha)}_{(\beta)} &= {}^4\omega^{(\alpha)}_{(\gamma)(\beta)} {}^4\theta^{(\gamma)} = {}^4\omega^{(\alpha)}_{\mu(\beta)} dx^\mu, \\
{}^4\omega_{(\alpha)(\gamma)(\beta)} &= {}^4\eta_{(\alpha)(\delta)} {}^4E^{(\delta)}_{\nu} {}^4E^\mu_{(\gamma)} {}^4\nabla_\mu {}^4E^\nu_{(\beta)} = {}^4\eta_{(\alpha)(\delta)} {}^4\omega^{(\delta)}_{(\gamma)(\beta)}, \\
{}^4\omega^{(\alpha)}_{\mu(\beta)} &= {}^4\omega^{(\alpha)}_{(\gamma)(\beta)} {}^4E^\mu_{(\gamma)} = {}^4E^{(\alpha)}_{\nu} {}^4\nabla_\mu {}^4E^\nu_{(\beta)} = {}^4E^{(\alpha)}_{\nu} [\partial_\mu {}^4E^\nu_{(\beta)} + {}^4\Gamma^\nu_{\mu\rho} {}^4E^\rho_{(\beta)}], \\
\Rightarrow {}^4\Gamma^\mu_{\rho\sigma} &= \frac{1}{2} [{}^4E^{(\beta)}_{\sigma} ({}^4E^\mu_{(\alpha)} {}^4E^{(\gamma)}_{\rho} {}^4\omega^{(\alpha)}_{(\gamma)(\beta)} - \partial_\rho {}^4E^\mu_{(\beta)}) + \\
&\quad + {}^4E^{(\beta)}_{\rho} ({}^4E^\mu_{(\alpha)} {}^4E^{(\gamma)}_{\sigma} {}^4\omega^{(\alpha)}_{(\gamma)(\beta)} - \partial_\sigma {}^4E^\mu_{(\beta)})], \tag{A15}
\end{aligned}$$

and the metric compatibility ${}^4\nabla_\rho {}^4g_{\mu\nu} = 0$ becomes the following condition

$${}^4\omega_{(\alpha)(\beta)} = {}^4\eta_{(\alpha)(\delta)} {}^4\omega^{(\delta)}_{(\beta)} = {}^4\eta_{(\alpha)(\delta)} {}^4\omega^{(\delta)}_{(\gamma)(\beta)} {}^4\theta^{(\gamma)} = {}^4\omega_{(\alpha)(\gamma)(\beta)} {}^4\theta^{(\gamma)} = -{}^4\omega_{(\beta)(\alpha)} \tag{A16}$$

or ${}^4\omega_{(\alpha)(\gamma)(\beta)} = -{}^4\omega_{(\beta)(\gamma)(\alpha)}$ ⁸⁵

Given a vector ${}^4V^\mu = {}^4V^{(\alpha)} {}^4E^\mu_{(\alpha)}$ and a covector ${}^4\omega_\mu = {}^4\omega_{(\alpha)} {}^4E^\mu_{(\alpha)}$, we define the covariant derivative of the components ${}^4V^{(\alpha)}$ and ${}^4\omega_{(\alpha)}$ as ${}^4\nabla_\nu {}^4V^\mu = {}^4V^\mu_{;\nu} \equiv [{}^4\nabla_\nu {}^4V^{(\alpha)}] {}^4E^\mu_{(\alpha)} = {}^4V^{(\alpha)}_{;\nu} {}^4E^\mu_{(\alpha)}$ and ${}^4\nabla_\nu {}^4\omega_\mu = {}^4\omega_{\mu;\nu} \equiv [{}^4\nabla_\nu {}^4\omega_{(\alpha)}] {}^4E^\mu_{(\alpha)} = {}^4\omega_{(\alpha);\nu} {}^4E^\mu_{(\alpha)}$, so that

$$\begin{aligned}
{}^4V^\mu_{;\nu} &= \partial_\nu {}^4V^{(\alpha)} {}^4E^\mu_{(\alpha)} + {}^4V^{(\alpha)} {}^4E^\mu_{(\alpha);\nu}, \\
&\Rightarrow {}^4V^{(\alpha)}_{;\nu} = \partial_\nu {}^4V^{(\alpha)} + {}^4\omega^{(\alpha)}_{\nu(\beta)} {}^4V^{(\beta)}, \\
{}^4\omega_{\mu;\nu} &= \partial_\nu {}^4\omega_{(\alpha)} {}^4E^\mu_{(\alpha)} + {}^4\omega_{(\alpha)} {}^4E^\mu_{(\alpha);\nu}, \\
&\Rightarrow {}^4\omega_{(\alpha);\nu} = \partial_\nu {}^4\omega_{(\alpha)} - {}^4\omega_{(\beta)} {}^4\omega^{(\beta)}_{\nu(\alpha)}. \tag{A17}
\end{aligned}$$

Therefore, for the *internal tensors* ${}^4T^{(\alpha)\dots}_{(\beta)\dots}$, the spin connection ${}^4\omega^{(\alpha)}_{\mu(\beta)}$ is a gauge potential associated with a gauge group $SO(3,1)$. For internal vectors ${}^4V^{(\alpha)}$ at $p \in M^4$ the cotetrads ${}^4E^\mu_{(\alpha)}$ realize a *soldering* of this internal vector space at p with the tangent space $T_p M^4$: ${}^4V^{(\alpha)} = {}^4E^\mu_{(\alpha)} {}^4V^\mu$. For tensors with mixed world and internal indices, like tetrads and cotetrads, we could define a generalized covariant derivative acting on both types of indices ${}^4\tilde{\nabla}_\nu {}^4E^\mu_{(\alpha)} = \partial_\nu {}^4E^\mu_{(\alpha)} + {}^4\Gamma^\mu_{\nu\rho} {}^4E^\rho_{(\alpha)} - {}^4E^\mu_{(\beta)} {}^4\omega^{(\beta)}_{\nu(\alpha)}$: then ${}^4\nabla_\nu {}^4V^\mu = {}^4\nabla_\nu {}^4V^{(\alpha)} {}^4E^\mu_{(\alpha)} + {}^4V^{(\alpha)} {}^4\tilde{\nabla}_\nu {}^4E^\mu_{(\alpha)} \equiv {}^4\nabla_\nu {}^4V^{(\alpha)} {}^4E^\mu_{(\alpha)}$ implies ${}^4\tilde{\nabla}_\nu {}^4E^\mu_{(\alpha)} = 0$ (or ${}^4\nabla_\nu {}^4E^\mu_{(\alpha)} = {}^4E^\mu_{(\beta)} {}^4\omega^{(\beta)}_{\nu(\alpha)}$) which is nothing else than the definition (A15) of the spin connection ${}^4\omega^{(\alpha)}_{\mu(\beta)}$.

We have

$$\begin{aligned}
[{}^4E_{(\alpha)}, {}^4E_{(\beta)}] &= c_{(\alpha)(\beta)}^{(\gamma)} {}^4E_{(\gamma)} = {}^4\nabla_{E_{(\alpha)}} {}^4E_{(\beta)} - {}^4\nabla_{E_{(\beta)}} {}^4E_{(\alpha)} = \\
&= ({}^4\omega^{(\gamma)}_{(\alpha)(\beta)} - {}^4\omega^{(\gamma)}_{(\beta)(\alpha)}) {}^4E_{(\gamma)}, \tag{A18}
\end{aligned}$$

$$\begin{aligned}
{}^4\Omega_{\mu\nu}^{(\alpha)} &= {}^4E^{(\gamma)}_{\mu} {}^4E^{(\delta)}_{\nu} {}^4\Omega^{(\alpha)}_{(\beta)(\gamma)(\delta)} = {}^4R^\rho{}_{\sigma\mu\nu} {}^4E^{(\alpha)}_{\rho} {}^4E^\sigma_{(\beta)} = \\
&= \partial_\mu {}^4\omega^{(\alpha)}_{\nu(\beta)} - \partial_\nu {}^4\omega^{(\alpha)}_{\mu(\beta)} + {}^4\omega^{(\alpha)}_{\mu(\gamma)} {}^4\omega^{(\gamma)}_{\nu(\beta)} - {}^4\omega^{(\alpha)}_{\nu(\gamma)} {}^4\omega^{(\gamma)}_{\mu(\beta)}, \\
{}^4R^\alpha{}_{\beta\mu\nu} &= {}^4E^\alpha_{(\gamma)} {}^4E^{(\delta)}_{\beta} {}^4\Omega_{\mu\nu}^{(\gamma)}{}_{(\delta)}. \tag{A19}
\end{aligned}$$

⁸⁵ ${}^4\omega_{(\alpha)(\gamma)(\beta)}$ are the *Ricci rotation coefficients*, only 24 of which are independent.

Let us remark that Eqs.(A14) and (A15) imply ${}^4\Gamma_{\mu\nu}^\rho = {}^4\Delta_{\mu\nu}^\rho + {}^4\omega_{\mu\nu}^\rho$ with ${}^4\omega_{\mu\nu}^\rho = {}^4E_{(\alpha)}^\rho {}^4E_\nu^{(\beta)} {}^4\omega_{\mu(\beta)}^{(\alpha)}$ and ${}^4\Delta_{\mu\nu}^\rho = {}^4E_{(\alpha)}^\rho \partial_\mu {}^4E_\nu^{(\alpha)}$; the Levi-Civita connection (i.e. the Christoffel symbols) turn out to be decomposed in a flat connection ${}^4\Delta_{\mu\nu}^\rho$ (it produces zero Riemann tensor as was already known to Einstein [75]) and in a tensor, like in the Yang-Mills case [8].

3. Triads and Cotriads on Σ_τ .

On Σ_τ with local coordinate system $\{\sigma^r\}$ and Riemannian metric ${}^3g_{rs}$ of signature $(+++)$ we can introduce orthonormal frames (*triads*) ${}^3e_{(a)} = {}^3e_{(a)}^r \frac{\partial}{\partial \sigma^r}$, $a=1,2,3$, and coframes (*cotriads*) ${}^3\theta^{(a)} = {}^3e_r^{(a)} d\sigma^r$ satisfying

$$\begin{aligned} {}^3e_{(a)}^r {}^3g_{rs} {}^3e_{(b)}^s &= \delta_{(a)(b)}, & {}^3e_r^{(a)} {}^3g^{rs} {}^3e_s^{(b)} &= \delta^{(a)(b)}, \\ {}^3e_{(a)}^r \delta^{(a)(b)} {}^3e_{(b)}^s &= {}^3g^{rs}, & {}^3e_r^{(a)} \delta_{(a)(b)} {}^3e_s^{(b)} &= {}^3g_{rs}. \end{aligned} \quad (\text{A20})$$

and consider the orthonormal frame bundle $F(\Sigma_\tau)$ over Σ_τ with structure group $\text{SO}(3)$. See Ref. [69] for geometrical properties of triads.

The 3-dimensional spin connection 1-form ${}^3\omega_{r(b)}^{(a)} d\sigma^r$ is

$$\begin{aligned} {}^3\omega_{r(b)}^{(a)} &= {}^3\omega_{(c)(b)}^{(a)} {}^3e_r^{(c)} = {}^3e_s^{(a)} {}^3\nabla_r {}^3e_{(b)}^s = \\ &= {}^3e_s^{(a)} {}^3e_{(b)|r}^s = {}^3e_s^{(a)} [\partial_r {}^3e_{(b)}^s + {}^3\Gamma_{ru}^s {}^3e_{(b)}^u], \\ {}^3\omega_{(a)(b)} &= \delta_{(a)(c)} {}^3\omega_{r(b)}^{(c)} d\sigma^r = -{}^3\omega_{(b)(a)}, & {}^3\omega_{r(a)} &= \frac{1}{2} \epsilon_{(a)(b)(c)} {}^3\omega_{r(b)(c)}, \\ {}^3\omega_{r(a)(b)} &= \epsilon_{(a)(b)(c)} {}^3\omega_{r(c)} = [\hat{R}^{(c)} {}^3\omega_{r(c)}]_{(a)(b)} = [{}^3\omega_r]_{(a)(b)}, \\ [{}^3e_{(a)}, {}^3e_{(b)}] &= ({}^3\omega_{(a)(b)}^{(c)} - {}^3\omega_{(b)(a)}^{(c)}) {}^3e_{(c)}, \end{aligned} \quad (\text{A21})$$

where $\epsilon_{(a)(b)(c)}$ is the standard Euclidean antisymmetric tensor and $(\hat{R}^{(c)})_{(a)(b)} = \epsilon_{(a)(b)(c)}$ is the adjoint representation of $\text{SO}(3)$ generators.

Given vectors and covectors ${}^3V^r = {}^3V^{(a)} {}^3e_{(a)}^r$, ${}^3V_r = {}^3V_{(a)} {}^3e_r^{(a)}$, we have ⁸⁶

$$\begin{aligned} {}^3\nabla_s {}^3V^r &= {}^3V^r|_s \equiv {}^3V_{|s}^{(a)} {}^3e_{(a)}^r, \\ &\Rightarrow {}^3V_{|s}^{(a)} = \partial_s {}^3V^{(a)} + {}^3\omega_{s(b)}^{(a)} {}^3V^{(b)} = \partial_s {}^3V^{(a)} + \delta^{(a)(c)} \epsilon_{(c)(b)(d)} {}^3\omega_{s(d)} {}^3V^{(b)}, \\ {}^3\nabla_s {}^3V_r &= {}^3V_r|_s = {}^3V_{(a)|s} {}^3e_r^{(a)}, \\ &\Rightarrow {}^3V_{(a)|s} = \partial_s {}^3V_{(a)} - {}^3V_{(b)} {}^3\omega_{s(a)}^{(b)} = \partial_s {}^3V_{(a)} - {}^3V_{(b)} \delta^{(b)(c)} \epsilon_{(c)(a)(d)} {}^3\omega_{s(d)}. \end{aligned} \quad (\text{A22})$$

For the field strength and the curvature tensors we have

⁸⁶Remember that ${}^3\nabla_s {}^3e_{(a)}^r = {}^3e_{(b)}^r {}^3\omega_{s(a)}^{(b)}$.

$$\begin{aligned}
{}^3\Omega^{(a)}{}_{(b)(c)(d)} &= {}^3e_{(c)}({}^3\omega_{(d)(b)}^{(a)}) - {}^3e_{(d)}({}^3\omega_{(c)(b)}^{(a)}) + \\
&\quad + {}^3\omega_{(d)(b)}^{(n)} {}^3\omega_{(c)(n)}^{(a)} - {}^3\omega_{(c)(b)}^{(n)} {}^3\omega_{(d)(n)}^{(a)} - ({}^3\omega_{(c)(d)}^{(n)} - {}^3\omega_{(d)(c)}^{(n)}) {}^3\omega_{(a)(b)}^{(a)} = \\
&= {}^3e_r^{(a)} {}^3R_{stw}^r {}^3e_{(b)}^s {}^3e_{(c)}^t {}^3e_{(d)}^w,
\end{aligned}$$

$$\begin{aligned}
{}^3\Omega_{rs}{}^{(a)}{}_{(b)} &= {}^3e_r^{(c)} {}^3e_s^{(d)} {}^3\Omega^{(a)}{}_{(b)(c)(d)} = {}^3R_{wrs}^t {}^3e_t^{(a)} {}^3e_{(b)}^w = \\
&= \partial_r {}^3\omega_{s(b)}^{(a)} - \partial_s {}^3\omega_{r(b)}^{(a)} + {}^3\omega_{r(c)}^{(a)} {}^3\omega_{s(b)}^{(c)} - {}^3\omega_{s(c)}^{(a)} {}^3\omega_{r(b)}^{(c)} = \\
&= \delta^{(a)(c)} {}^3\Omega_{rs(c)(b)} = \delta^{(a)(c)} \epsilon_{(c)(b)(d)} {}^3\Omega_{rs(d)},
\end{aligned}$$

$${}^3\Omega_{rs(a)} = \frac{1}{2} \epsilon_{(a)(b)(c)} {}^3\Omega_{rs(b)(c)} = \partial_r {}^3\omega_{s(a)} - \partial_s {}^3\omega_{r(a)} - \epsilon_{(a)(b)(c)} {}^3\omega_{r(b)} {}^3\omega_{s(c)},$$

$$\begin{aligned}
{}^3R_{stw}^r &= \epsilon_{(a)(b)(c)} {}^3e_{(a)}^r \delta_{(b)(n)} {}^3e_s^{(n)} {}^3\Omega_{tw(c)}, \\
{}^3R_{rs} &= \epsilon_{(a)(b)(c)} {}^3e_{(a)}^u \delta_{(b)(n)} {}^3e_r^{(n)} {}^3\Omega_{us(c)}, \\
{}^3R &= \epsilon_{(a)(b)(c)} {}^3e_{(a)}^r {}^3e_{(b)}^s {}^3\Omega_{rs(c)}.
\end{aligned} \tag{A23}$$

The first Bianchi identity (A2) ${}^3R_{rsu}^t + {}^3R_{sur}^t + {}^3R_{urs}^t \equiv 0$ implies the cyclic identity ${}^3\Omega_{rs(a)} {}^3e_{(a)}^s \equiv 0$.

Under local SO(3) rotations R [$R^{-1} = R^T$] we have

$$\begin{aligned}
{}^3\omega_{r(b)}^{(a)} &\mapsto [R {}^3\omega_r R^T - R \partial_r R^T]^{(a)}{}_{(b)}, \\
{}^3\Omega_{rs}{}^{(a)}{}_{(b)} &\mapsto [R {}^3\Omega_{rs} R^T]^{(a)}{}_{(b)}.
\end{aligned} \tag{A24}$$

Since the flat metric $\delta_{(a)(b)}$ has signature $(+++)$, we have ${}^3V^{(a)} = \delta^{(a)(b)} {}^3V_{(b)} = {}^3V_{(a)}$ and one can simplify the notations by using only lower (a) indices: ${}^3e_r^{(a)} = {}^3e_{(a)r}$. For instance, we have

$$\begin{aligned}
{}^3\Gamma_{rs}^u &= {}^3\Gamma_{sr}^u = \frac{1}{2} {}^3e_{(a)}^u [\partial_r {}^3e_{(a)s} + \partial_s {}^3e_{(a)r} + \\
&\quad + {}^3e_{(a)}^v ({}^3e_{(b)r} (\partial_s {}^3e_{(b)v} - \partial_v {}^3e_{(b)s}) + {}^3e_{(b)s} (\partial_r {}^3e_{(b)v} - \partial_v {}^3e_{(b)r}))] = \\
&= \frac{1}{2} \epsilon_{(a)(b)(c)} {}^3e_{(a)}^u ({}^3e_{(b)r} {}^3\omega_{s(c)} + {}^3e_{(b)s} {}^3\omega_{r(c)}) - \frac{1}{2} ({}^3e_{(a)r} \partial_s {}^3e_{(a)}^u + {}^3e_{(a)s} \partial_r {}^3e_{(a)}^u), \\
{}^3\omega_{r(a)(b)} &= -{}^3\omega_{r(b)(a)} = \frac{1}{2} [{}^3e_{(a)}^s (\partial_r {}^3e_{(b)s} - \partial_s {}^3e_{(b)r}) + \\
&\quad + {}^3e_{(b)}^s (\partial_s {}^3e_{(a)r} - \partial_r {}^3e_{(a)s}) + {}^3e_{(a)}^u {}^3e_{(b)}^v {}^3e_{(c)r} (\partial_v {}^3e_{(c)u} - \partial_u {}^3e_{(c)v})] = \\
&= \frac{1}{2} [{}^3e_{(a)u} \partial_r {}^3e_{(b)}^u - {}^3e_{(b)u} \partial_r {}^3e_{(a)}^u + {}^3\Gamma_{rs}^u ({}^3e_{(a)u} {}^3e_{(b)}^s - {}^3e_{(b)u} {}^3e_{(a)}^s)], \\
{}^3\omega_{r(a)} &= \frac{1}{2} \epsilon_{(a)(b)(c)} [{}^3e_{(b)}^u (\partial_r {}^3e_{(c)u} - \partial_u {}^3e_{(c)r}) + \\
&\quad + \frac{1}{2} {}^3e_{(b)}^u {}^3e_{(c)}^v {}^3e_{(d)r} (\partial_v {}^3e_{(d)u} - \partial_u {}^3e_{(d)v})], \\
{}^3\Omega_{rs(a)} &= \frac{1}{2} \epsilon_{(a)(b)(c)} [\partial_r {}^3e_{(b)}^u \partial_s {}^3e_{(c)u} - \partial_s {}^3e_{(b)}^u \partial_r {}^3e_{(c)u} + \\
&\quad + {}^3e_{(b)}^u (\partial_u \partial_s {}^3e_{(c)r} - \partial_u \partial_r {}^3e_{(c)s}) +
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left({}^3e_{(b)}^u {}^3e_{(c)}^v (\partial_r {}^3e_{(d)s} - \partial_s {}^3e_{(d)r}) (\partial_v {}^3e_{(d)u} - \partial_u {}^3e_{(d)v}) + \right. \\
& + \left. ({}^3e_{(d)s} \partial_r - {}^3e_{(d)r} \partial_s) [{}^3e_{(b)}^u {}^3e_{(c)}^v (\partial_v {}^3e_{(d)u} - \partial_u {}^3e_{(d)v})] \right) - \\
& - \frac{1}{8} [\delta_{(a)(b_1)} \epsilon_{(c_1)(c_2)(b_2)} + \delta_{(a)(b_2)} \epsilon_{(c_1)(c_2)(b_1)} + \delta_{(a)(c_1)} \epsilon_{(b_1)(b_2)(c_2)} + \delta_{(a)(c_2)} \epsilon_{(b_1)(b_2)(c_1)}] \times \\
& \quad {}^3e_{(b_1)}^{u_1} {}^3e_{(b_2)}^{u_2} [(\partial_r {}^3e_{(c_1)u_1} - \partial_{u_1} {}^3e_{(c_1)r}) (\partial_s {}^3e_{(c_2)u_2} - \partial_{u_2} {}^3e_{(c_2)s}) + \\
& + \frac{1}{2} ({}^3e_{(c_2)}^{v_2} {}^3e_{(d)s} (\partial_r {}^3e_{(c_1)u_1} - \partial_{u_1} {}^3e_{(c_1)r}) (\partial_{v_2} {}^3e_{(d)u_2} - \partial_{u_2} {}^3e_{(d)v_2}) + \\
& + {}^3e_{(c_1)}^{v_1} {}^3e_{(d)r} (\partial_s {}^3e_{(c_2)u_2} - \partial_{u_2} {}^3e_{(c_2)s}) (\partial_{v_1} {}^3e_{(d)u_1} - \partial_{u_1} {}^3e_{(d)v_1})) + \\
& + \frac{1}{4} {}^3e_{(c_1)}^{v_1} {}^3e_{(c_2)}^{v_2} {}^3e_{(d_1)r} {}^3e_{(d_2)s} (\partial_{v_1} {}^3e_{(d_1)u_1} - \partial_{u_1} {}^3e_{(d_1)v_1}) (\partial_{v_2} {}^3e_{(d_2)u_2} - \partial_{u_2} {}^3e_{(d_2)v_2})], \\
& {}^3\Omega_{rs(a)(b)} = \epsilon_{(a)(b)(c)} {}^3\Omega_{rs(c)}, \\
& {}^3R_{rsuv} = \epsilon_{(a)(b)(c)} {}^3e_{(a)r} {}^3e_{(b)s} {}^3\Omega_{uv(c)}, \\
& {}^3R_{rs} = \frac{1}{2} \epsilon_{(a)(b)(c)} {}^3e_{(a)}^u [{}^3e_{(b)r} {}^3\Omega_{us(c)} + {}^3e_{(b)s} {}^3\Omega_{ur(c)}], \\
& {}^3R = \epsilon_{(a)(b)(c)} {}^3e_{(a)}^r {}^3e_{(b)}^s {}^3\Omega_{rs(c)}. \tag{A25}
\end{aligned}$$

4. Action Principles.

Let us finish this Appendix with a review of some action principles used for general relativity. In *metric gravity*, one uses the generally covariant *Hilbert action* depending on the 4-metric and its first and second derivatives ⁸⁷

$$S_H = \frac{c^3}{16\pi G} \int_U d^4x \sqrt{^4g} {}^4R = \int_U d^4x \mathcal{L}_H. \tag{A26}$$

The variation of S_H is ($d^3\Sigma_\gamma = d^3\Sigma l_\gamma$)

$$\begin{aligned}
\delta S_H &= \delta S_E + \Sigma_H = -\frac{c^3}{16\pi G} \int_U d^4x \sqrt{^4g} {}^4G^{\mu\nu} \delta^4 g_{\mu\nu} + \Sigma_H, \\
\Sigma_H &= \frac{c^3}{16\pi G} \int_{\partial U} d^3\Sigma_\gamma \sqrt{^4g} ({}^4g^{\mu\nu} \delta_\delta^\gamma - {}^4g^{\mu\gamma} \delta_\delta^\nu) \delta^4 \Gamma_{\mu\nu}^\delta = \\
&= \frac{c^3}{8\pi G} \int_{\partial U} d^3\Sigma \sqrt{^3\gamma} \delta^3 K, \\
\delta^4 \Gamma_{\mu\nu}^\delta &= \frac{1}{2} g^{\delta\beta} [{}^4\nabla_\mu \delta^4 g_{\beta\nu} + {}^4\nabla_\nu \delta^4 g_{\beta\mu} - {}^4\nabla_\beta \delta^4 g_{\mu\nu}]. \tag{A27}
\end{aligned}$$

where ${}^3\gamma_{\mu\nu}$ is the metric induced on ∂U and l_μ is the outer unit covariant normal to ∂U . The trace of the extrinsic curvature ${}^3K_{\mu\nu}$ of ∂U is ${}^3K = -l^\mu{}_{;\mu}$. The surface term Σ_H

⁸⁷G is Newton gravitational constant; $U \subset M^4$ is a subset of spacetime; we use units with $x^0 = ct$.

takes care of the second derivatives of the 4-metric and to get Einstein equations ${}^4G_{\mu\nu} = {}^4R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}{}^4R \stackrel{\circ}{=} 0$ one must take constant certain normal derivatives of the 4-metric on the boundary of U [$\mathcal{L}_l({}^4g_{\mu\nu} - l_\mu l_\nu) = 0$] to have $\delta S_H = 0$ [76].

The term δS_E in Eq.(A25) means the variation of the action S_E , which is the (not generally covariant) *Einstein action* depending only on the 4-metric and its first derivatives

$$\begin{aligned} S_E &= \int_U d^4x \mathcal{L}_E = \frac{c^3}{16\pi G} \int_U d^4x \sqrt{{}^4g} {}^4g^{\mu\nu} ({}^4\Gamma_{\nu\lambda}^\rho {}^4\Gamma_{\rho\mu}^\lambda - {}^4\Gamma_{\lambda\rho}^\lambda {}^4\Gamma_{\mu\nu}^\rho) = \\ &= S_H - \frac{c^3}{16\pi G} \int_U d^4x \partial_\lambda [\sqrt{{}^4g} ({}^4g^{\mu\nu} {}^4\Gamma_{\mu\nu}^\lambda - {}^4g^{\lambda\mu} {}^4\Gamma_{\rho\mu}^\rho)], \\ \delta S_E &= \frac{c^3}{16\pi G} \int_U d^4x \left(\frac{\partial \mathcal{L}_E}{\partial {}^4g^{\mu\nu}} - \partial_\rho \frac{\partial \mathcal{L}_E}{\partial \partial_\rho {}^4g^{\mu\nu}} \right) \delta {}^4g^{\mu\nu} = -\frac{c^3}{16\pi G} \int_U d^4x \sqrt{{}^4g} {}^4G_{\mu\nu} \delta {}^4g^{\mu\nu}. \quad (\text{A28}) \end{aligned}$$

We shall not consider the first-order *Palatini action*; see for instance Ref. [66], where there is also a review of the variational principles of the connection-dependent formulations of general relativity.

In Ref. [76] (see also Ref. [73]), it is shown that the *DeWitt-ADM action* [64,31] for a 3+1 decomposition of M^4 can be obtained from S_H in the following way⁸⁹

$$\begin{aligned} S_H &= S_{ADM} + \Sigma_{ADM}, \\ S_{ADM} &= -\epsilon \frac{c^3}{16\pi G} \int_U d^4x \sqrt{{}^4g} [{}^3R + {}^3K_{\mu\nu} {}^3K^{\mu\nu} - ({}^3K)^2], \\ \Sigma_{ADM} &= -\epsilon \frac{c^3}{8\pi G} \int d^4x \partial_\alpha [\sqrt{{}^4g} ({}^3K l^\alpha + l^\beta l^\alpha_{;\beta})] = \\ &= -\epsilon \frac{c^3}{8\pi G} \left[\int_S d^3\sigma [\sqrt{\gamma} {}^3K](\tau, \vec{\sigma})|_{\tau_i}^{\tau_f} + \right. \\ &\quad \left. + \int_{\tau_i}^{\tau_f} d\tau \int_{\partial S} d^2\Sigma^r [{}^3\nabla_r (\sqrt{\gamma} N) - {}^3K N_r](\tau, \vec{\sigma}) \right], \\ \delta S_{ADM} &= -\epsilon \frac{c^3}{16\pi G} \int d\tau d^3\sigma \sqrt{\gamma} [2 {}^4\bar{G}_{ll} \delta N + {}^4\bar{G}_l^r \delta N_r - {}^4\bar{G}^{rs} \delta {}^3g_{rs}](\tau, \vec{\sigma}) + \\ &\quad + \delta S_{ADM}|_{{}^4G_{\mu\nu}=0} - \epsilon \int_{\tau_i}^{\tau_f} d\tau \int_{\partial U} d^3\Sigma_r [N_{|s} \delta {}^3g^{rs} - N \delta {}^3g^{rs}_{|s}](\tau, \vec{\sigma}), \\ \delta S_{ADM}|_{{}^4G_{\mu\nu}=0} &= -\epsilon \frac{c^3}{16\pi G} \int_{\partial U} d^3\sigma {}^3\tilde{\Pi}^{\mu\nu} \delta {}^3\gamma_{\mu\nu}, \\ {}^3\tilde{\Pi}^{\mu\nu} &= \sqrt{\gamma} ({}^3K^{\mu\nu} - {}^3g^{\mu\nu} {}^3K) = \frac{16\pi G}{c^3} \epsilon \hat{b}_r^\mu \hat{b}_s^\nu {}^3\tilde{\Pi}^{rs}, \quad (\text{A29}) \end{aligned}$$

so that $\delta S_{ADM} = 0$ gives ${}^4G_{\mu\nu} \stackrel{\circ}{=} 0$ if one holds fixed the intrinsic 3-metric ${}^3\gamma_{\mu\nu}$ on the

⁸⁸ $\delta S_E = 0$ gives ${}^4G_{\mu\nu} \stackrel{\circ}{=} 0$ if ${}^4g_{\mu\nu}$ is held fixed on ∂U .

⁸⁹ $\sqrt{{}^4g} {}^4R = -\epsilon \sqrt{{}^4g} ({}^3R + {}^3K_{\mu\nu} {}^3K^{\mu\nu} - ({}^3K)^2) - 2\epsilon \partial_\lambda (\sqrt{{}^4g} ({}^3K l^\lambda + a^\lambda))$, with a^λ the 4-acceleration ($l^\mu a_\mu = 0$); the 4-volume U is $[\tau_f, \tau_i] \times S$.

boundary⁹⁰. This action is not generally covariant, but it is quasi-invariant under the 8 types of gauge transformations generated by the ADM first class constraints (see Appendix A of Ref. [77]). As shown in Refs. [78,32,76,79] in this way one obtains a well defined gravitational energy. However, in so doing one still neglects some boundary terms. Following Ref. [79], let us assume that, given a subset $U \subset M^4$ of spacetime, ∂U consists of two slices, Σ_{τ_i} (the initial one) and Σ_{τ_f} (the final one) with outer normals $-l^\mu(\tau_i, \vec{\sigma})$ and $l^\mu(\tau_f, \vec{\sigma})$ respectively, and of a surface S_∞ near space infinity with outer unit (spacelike) normal $n^\mu(\tau, \vec{\sigma})$ tangent to the slices⁹¹. The 3-surface S_∞ is foliated by a family of 2-surfaces $S_{\tau,\infty}^2$ coming from its intersection with the slices Σ_τ ⁹². The vector $b_\tau^\mu = z_\tau^\mu = Nl^\mu + N^r b_\tau^\mu$ is not in general tangent to S_∞ . It is assumed that there are no inner boundaries (see Ref. [79] for their treatment), so that the slices Σ_τ do not intersect and are complete. This does not rule out the existence of horizons, but it implies that, if horizons form, one continues to evolve the spacetime inside the horizon as well as outside. Then, in Ref. [79] it is shown that one gets⁹³

$$\begin{aligned}\Sigma_{ADM} &= -\epsilon \frac{c^3}{8\pi G} \left[\int_{\Sigma_{\tau_f}} d^3\Sigma - \int_{\Sigma_{\tau_i}} d^3\Sigma \right] N \sqrt{\gamma}^3 K = \\ &= -\epsilon \frac{c^3}{8\pi G} \int_{\tau_i}^{\tau_f} d\tau N_{(as)}(\tau) \int_{S_{\tau,\infty}^2} d^2\Sigma \sqrt{\gamma}^2 K.\end{aligned}\tag{A30}$$

In Einstein metric gravity the gravitational field, described by the 4-metric ${}^4g_{\mu\nu}$ depends on 2, and not 10, physical degrees of freedom in each point; this is not explicitly evident if one starts with the Hilbert action, which is invariant under $Diff M^4$, a group with only four generators. Instead in ADM canonical gravity (see Section IV) there are in each point 20 canonical variables and 8 first class constraints, implying the determination of 8 canonical variables and the arbitrariness of the 8 conjugate ones. At the Lagrangian level, only 6 of the ten Einstein equations are independent, due to the contracted Bianchi identities, so that four components of the metric tensor ${}^4g_{\mu\nu}$ (the lapse and shift functions) are arbitrary not being determined by the equation of motion. Moreover, the four combinations ${}^4\bar{G}_{ll} \doteq 0$, ${}^4\bar{G}_{lr} \doteq 0$, of the Einstein equations do not depend on the second time derivatives or accelerations (they are restrictions on the Cauchy data and become the secondary first class constraints of the ADM canonical theory): the general theory [4] implies that four generalized velocities (and therefore other four components of the metric) inherit the arbitrariness of the lapse and shift functions. Only two combinations of the Einstein equations depend on the accelerations

^{90,3} $\tilde{\Pi}^{\mu\nu}$ is the ADM momentum with world indices, whose form in a 3+1 splitting is given in Section IV.

⁹¹So that the normal $l^\mu(\tau, \vec{\sigma})$ to every slice is asymptotically tangent to S_∞ .

⁹²Therefore, asymptotically $l^\mu(\tau, \vec{\sigma})$ is normal to the corresponding $S_{\tau,\infty}^2$.

^{93,2} K the trace of the 2-dimensional extrinsic curvature of the 2-surface $S_{\tau,\infty}^2 = S_\infty \cap \Sigma_\tau$; to get this result one assumes that the lapse function $N(\tau, \vec{\sigma})$ on Σ_τ tends asymptotically to a function $N_{(as)}(\tau)$ and that the term on ∂S vanishes due to the boundary conditions.

(second time derivatives) of the two (non tensorial) independent degrees of freedom of the gravitational field and are genuine equations of motion. Therefore, the ten components of every 4-metric ${}^4g_{\mu\nu}$, compatible with the Cauchy data, depend on 8 arbitrary functions not determined by the Einstein equations.

Instead, in *tetrad gravity* [16–21, 23–27], in which ${}^4g_{\mu\nu}$ is no more the independent variable, the new independent 16 variables are a set of cotetrads ${}^4E_{\mu}^{(\alpha)}$ so that ${}^4g_{\mu\nu} = {}^4E_{\mu}^{(\alpha)} {}^4\eta_{(\alpha)(\beta)} {}^4E_{\nu}^{(\beta)}$. Tetrad gravity has not only the invariance under $Diff M^4$ but also under local Lorentz transformations on TM^4 [acting on the flat indices (α)]. An action principle with these local invariances is obtained by replacing the 4-metric in the Hilbert action S_H with its expression in terms of the cotetrads. The action acquires the form

$$S_{HT} = \frac{c^3}{16\pi G} \int_U d^4x {}^4\tilde{E} {}^4E_{(\alpha)}^{\mu} {}^4E_{(\beta)}^{\nu} {}^4\Omega_{\mu\nu}^{(\alpha)(\beta)}, \quad (A31)$$

where ${}^4\tilde{E} = \det({}^4E_{\mu}^{(\alpha)}) = \sqrt{{}^4g}$ and ${}^4\Omega_{\mu\nu}^{(\alpha)(\beta)}$ is the spin 4-field strength. One has

$$\begin{aligned} \delta S_{HT} = & \frac{c^3}{16\pi G} \int_U d^4x {}^4\tilde{E} {}^4G_{\mu\nu} {}^4E_{(\alpha)}^{\mu} {}^4\eta^{(\alpha)(\beta)} \delta {}^4E_{(\beta)}^{\nu} + \\ & + \frac{c^3}{8\pi G} \int_U d^4x \partial_{\mu} [{}^4\tilde{E} ({}^4E_{\nu}^{(\rho)} \delta({}^4g^{\mu\lambda} {}^4\nabla_{\lambda} {}^4E_{(\rho)}^{\nu}) - {}^4\eta^{(\rho)(\sigma)} {}^4E_{(\rho)}^{\nu} \delta({}^4\nabla_{\nu} {}^4E_{(\sigma)}^{\mu}))]. \end{aligned} \quad (A32)$$

Again $\delta S_{HT} = 0$ produces Einstein equations if complicated derivatives of the tetrads vanish at the boundary.

Tetrad gravity with action S_{HT} , in which the elementary natural Lagrangian object is the soldering or canonical one-form (or orthogonal coframe) $\theta^{(\alpha)} = {}^4E_{\mu}^{(\alpha)} dx^{\mu}$, is gauge invariant simultaneously under diffeomorphisms ($Diff M^4$) and Lorentz transformations [$SO(3,1)$]. Instead in phase space (see Section III) only two of the 16 components of the cotetrad ${}^4E_{\mu}^{(\alpha)}(x)$ are physical degrees of freedom in each point, since the 32 canonical variables present in each point are restricted by 14 first class constraints, so that the 16 components of a cotetrad compatible with the Cauchy data depend on 14 arbitrary functions not determined by the equation of motion.

In Ref. [21], by using ${}^4\tilde{E} {}^4E_{(\alpha)}^{\mu} {}^4E_{(\beta)}^{\nu} {}^4\Omega_{\mu\nu}^{(\alpha)(\beta)} = 2 {}^4\tilde{E} {}^4E_{(\alpha)}^{\mu} {}^4E_{(\beta)}^{\nu} [{}^4\omega_{\mu} {}^4\omega_{\nu} - {}^4\omega_{\nu} {}^4\omega_{\mu}]^{(\alpha)(\beta)} + 2 \partial_{\mu} ({}^4\tilde{E} {}^4E_{(\alpha)}^{\mu} {}^4E_{(\beta)}^{\nu} {}^4\omega_{\nu}^{(\alpha)(\beta)})$, the analogue of S_E , i.e. the (not locally Lorentz invariant, therefore not expressible only in terms of the 4-metric) *Charap action*, is defined as

$$S_C = -\frac{c^3}{8\pi G} \int_U d^4x {}^4\tilde{E} {}^4E_{(\alpha)}^{\mu} {}^4E_{(\beta)}^{\nu} ({}^4\omega_{\mu} {}^4\omega_{\nu} - {}^4\omega_{\nu} {}^4\omega_{\mu})^{(\alpha)(\beta)}. \quad (A33)$$

Its variation δS_C vanishes if $\delta {}^4E_{(\alpha)}^{\mu}$ vanish at the boundary and the Einstein equations hold. However its Hamiltonian formulation gives too complicated first class constraints to be solved.

Instead in Refs. [24–27] it was *implicitly* used the metric ADM action $S_{ADM}[{}^4g_{\mu\nu}]$ with the metric expressed in terms of cotetrads in the Schwinger time gauge [18] as independent Lagrangian variables $S_{ADMT}[{}^4E_{\mu}^{(\alpha)}]$. This is the action we shall study in this paper after having expressed arbitrary cotetrads in terms of Σ_{τ} -adapted ones in the next Section.

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